AXIOM SCHEMATA OF STRONG INFINITY IN AXIOMATIC SET THEORY

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1. Introduction. There are, in general, two main approaches to the introduction of strong infinity assertions to the Zermelo-Fraenkel set theory. The arithmetical approach starts with the regular ordinal numbers, continues with the weakly inaccessible numbers and goes on to the ρ_0-numbers of Mahlo [4], etc. The model-theoretic approach, with which we shall be concerned, introduces the strongly inaccessible numbers and leads to Tarski's axioms of [14] and [15]. As we shall see, even in the model-theoretic approach we can use methods for expressing strong assertions of infinity which are mainly arithmetical. Therefore we shall introduce strong axiom schemata of infinity by following Mahlo [4,5,6,]. Using the ideas of Montague in [7] we shall give those axiom schemata a purely model-theoretic form. Also the axiom schemata of replacement in conjunction with the axiom of infinity will be given a similar form, and thus the new axiom schemata will be seen to be natural continuations of the axiom schema of replacement and infinity.

A provisional notion of a standard model, introduced in § 2, will be basic for our discussion. However, in § 5 it is shown that this definition cannot serve as a general definition for the notion of a standard model.

2. Standard models of set theories. For the forthcoming discussion we need the notion of a standard model of a set theory. A general principle which distinguishes between standard and non-standard models of set theory is not yet known. Nevertheless, a notion of a standard model for various set theories will be given here, but this will serve only as an ad-hoc principle and we shall see later that its general application is not justified.

The Zermelo-Fraenkel set theory is generally formalized in the simple applied first-order functional calculus, since this is the most natural language for a set theory. In that formulation the Zermelo-Fraenkel set theory has an infinite number of axioms. From that formulation one passes directly to a formulation of the Zermelo-Fraenkel set theory by a finite number of axioms in the non-simple applied first-order functional calculus (we shall denote functional variables with $p, p_1, p_2, \cdots$). The axioms of extensionality, pairing, sum-set, power-set and infinity are as in [2]. The changed axioms are

The axiom of subsets $(x)(\exists y)(z)(z \in y \equiv : z \in x \,.\, p(z))$

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