SOME APPLICATIONS OF EXPANSION CONSTANTS

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1. For any metric space X (with distance function d) the *expansion constant* $E(X)$ of X is the greatest lower bound of real numbers μ which possess the following property ($S(x; \rho) = S_X(x; \rho) = \{y \in X; d(x, y) \leq \rho\}$ denotes the closed cell with center x and radius ρ):

For any family $\{S(x_\alpha; \rho_\alpha); \alpha \in A\}$ of pairwise intersecting cells in X,

$$\bigcap_{\alpha \in A} S(x_\alpha; \mu\rho_\alpha) \neq \phi .$$

If for every such family $\bigcap_{\alpha \in A} S(x_\alpha; E(X)\rho_\alpha) \neq \phi$, $E(X)$ is called *exact*.

The expansion constants of Minkowski spaces have been studied in [5]. In the present paper we deal (in § 2) with an application of the expansion constants to a problem on projections in Banach spaces; as corollaries we obtain Nachbin's [10] geometric characterization of Banach spaces with the Hahn-Banach extension property (§ 2) and Bohnenblust's [3] result on projections in Minkowski spaces, as well as some results which we believe to be new (§ 4). In § 3 we discuss the relation of expansion constants to a property of retractions in metric spaces, especially those convex in Menger's sense; as a corollary we obtain Aronszajn-Panitchpakdi's [2] characterization of spaces with the unlimited uniform extension property. Section 4 contains additional remarks and examples.

2. In order to apply expansion constants to projections in Banach spaces, it is convenient to introduce the notion of projection constants.

DEFINITION 1. For any normed space X the *projection constant* $p(X)$ is the greatest lower bound of real numbers μ which possess the following property: For any normed space Y which contains X as a subspace of deficiency 1, there exists a projection P of Y onto X such that $\|P\| \leq \mu$. If for any such Y there exists a projection of norm less than or equal to $p(X)$, the projection constant $p(X)$ is called *exact*.

(The projection constant $p(X)$ should not be confused with the projection constant $\mathscr{P}(X)$ studied in [6].)

We show now that if X is a normed space then $E(X)$ actually coincides with $p(X)$.

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