AREA AND NORMALITY

H. BUSEMANN AND E. G. STRAUS

1. Introduction. The simplest non-Riemannian a-dimensional area (concisely: a-area) is a translation invariant positive continuous measure (or area) defined on the a-dimensional linear subspaces, called a-flats, of an n-dimensional affine space A_n $(1 \leq a \leq n)$. Such areas have been studied by Wagner [15] and they are the subject of the present investigation which is in part related to Wagner's, but has no connection with the differential geometry of general area metrics persued principally in Japan by Kawaguchi, Iwamoto and others.

The simplest case, $a = 1$, is well known. In that case a segment with endpoints x, y has a translation invariant length $d(x, y)$. If the sphere $d(z, x) = 1$ (z fixed) has at x_0 a supporting $(n-1)$-flat (hyperplane) H_0 then H_0 is transversal to the 1-flat (line) L_0 through z and x_0, and L_0 is normal to H_0.

Therefore the existence of an $(n-1)$-flat transversal to a given line is equivalent to the convexity of the sphere $d(z, x) = 1$; which, in turn, is equivalent to the triangle inequality for $d(a, b)$, in other words, to the space being Minkowskian (normed linear).

If L_0 is normal to H_0 at x_0 then it is normal to every line L through x_0 in H_0 in the two-flat spanned by L_0 and L. A well-known theorem of Blaschke [2] states that for $n \geq 3$ normality between lines is symmetric only in euclidean space. However, as shown by Radon [13], this is not the case for $n = 2$.

Here we treat the analogous problems for arbitrary a, and then study the special case of Minkowski area.

We cannot give more than this vague hint without some definitions. Let $(x^1, \cdots, x^n)$ be affine coordinates of a point x in A^n with origin $z = (0, \cdots, 0)$. The *a-box* $[x_0, x_1, \cdots, x_a]$ consists of all points of the form $(1-\theta_i)x_0 + \sum_{1=1}^{a} \theta_i x_i$ where $0 \leq \theta_i, \leq 1$; and hence is a (possibly degenerate) parallelepiped.

An *a-area* assigns to every Borel[1] set M in an a-flat a measure $\alpha(M)$ which is invariant under the translations of A^n, and continuous; that is, $\alpha([x_0, \cdots, x_a])$ depends continuously on $x_0, \cdots, x_a$. The invariance under translation applied to sets in the same a-flat A yields at once that the measure in A is determined up to a factor depending on A. If we introduce an auxiliary euclidean metric

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[1]All sets considered will be Borel sets.