SEQUENCES OF COVERINGS

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1. Introduction. The metrisable spaces S for which S' (the set of limit points of S) is compact, can be characterized as those uniformisable spaces for which the finest uniformity (compatible with the topology) is metrisable (see [5], [1], where further characterizations are given). B. T. Levshenko has shown [4] that they also coincide with the regular spaces in which every point-finite covering[1] can be refined by one of a fixed sequence of point-finite coverings, and that "point-finite" can be replaced throughout by "star-finite" or "locally finite". We shall extend these results (Theorem 2) and obtain an analogue for uniform spaces (Theorem 3). The proofs depend on a criterion for metrisability (Theorem 1) which may be of independent interest since, though not really new in content, it is particularly simple in form.

NOTATION. If $\mathscr{U}$ is a covering of a space S, and $A \subset S$, the star $St(A, \mathscr{U})$ of A in $\mathscr{U}$ is $\bigcup \{U \mid U \in \mathscr{U}, A \cap U \neq \phi\}$. When A is a 1-point set (x), we abbreviate $St((x), \mathscr{U})$ to $St(x, \mathscr{U})$. The covering by the sets $St(U, \mathscr{U})$, $U \in \mathscr{U}$, is denoted by $St(\mathscr{U})$. A covering $\mathscr{U}$ will be called "almost discrete" if only finitely many pairs U, V of sets of $\mathscr{U}$ intersect; such a covering is clearly star-finite (in fact star-bounded) and so locally finite.

2. Metrisation criterion.

THEOREM 1. *A necessary and sufficient condition that a T_0 space S be metrisable is that S have a sequence of coverings $\mathscr{U}_n$, $n = 1, 2, \cdots$, such that, for each $x \in S$, the stars $St(G, \mathscr{U}_n)$ of the open sets $G \ni x$ form a basis for the neighborhoods of x.*

The condition is trivially necessary. To prove it sufficient, we observe first that S is developable—i.e., the stars $St(x, \mathscr{U}_n)$ form a basis for the neighborhoods of each $x \in S$. It follows that S is T_1; for if x, y are distinct points of S, one of them, say x, has a neighborhood $St(x, \mathscr{U}_n)$ not containing y, and then $St(y, \mathscr{U}_n)$ does not contain x. We next show that S is collectionwise normal (see [2]). We may assume that $\mathscr{U}_{n+1}$ refines $\mathscr{U}_n$ (by replacing each $\mathscr{U}_n$ by the "intersection" of the coverings $\mathscr{U}_1, \cdots, \mathscr{U}_n$). Let A_λ $(\lambda \in \Lambda)$ be a discrete collection of closed subsets of S, and for each n and λ put

$$H_{n\lambda} = \bigcup \{U \mid U \in \mathscr{U}_n,\ St(U, \mathscr{U}_n) \text{ meets } A_\lambda \text{ but not } A_\mu \text{ if } \mu \neq \lambda\}\ ,$$

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[1] Throughout this paper, "covering" means "open covering."