THE METRIZATION OF STATISTICAL METRIC SPACES

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In a previous paper on statistical metric spaces [3] it was shown that a statistical metric induces a natural topology for the space on which it is defined and that with this topology a large class of statistical metric (briefly, *SM*) spaces are Hausdorff spaces.

In this paper we show that this result (Theorem 7.2 of [3]) can be considerably generalized. In addition, as an immediate corollary of this generalization, we prove that with the given topology a large number of *SM* spaces are metrizable, i.e., that in numerous instances the existence of a statistical metric implies the existence of an ordinary metric.[1]

THEOREM 1.[2] *Let* $(S, \mathscr{F})$ *be a statistical metric space,* $\mathscr{U}$ *the two-parameter collection of subsets of* $S \times S$ *defined by*

$$\mathscr{U} = \{U(\varepsilon, \lambda); \varepsilon > 0,\ \lambda > 0\},$$

where

$$U(\varepsilon, \lambda) = \{(p, q);\ p, q \text{ in } S \text{ and } F_{pq}(\varepsilon) > 1 - \lambda\},$$

and T *a two-place function from* $[0, 1] \times [0, 1]$ *to* $[0, 1]$ *satisfying* $T(c, d) \geqq T(a, b)$ *for* $c \geqq a$, $d \geqq b$ *and* $\sup_{x<1} T(x, x) = 1$. *Suppose further that for all* p, q, r *in* S *and for all real numbers* x, y, *the Menger triangle inequality.*

$$(1) \qquad F_{pr}(x + y) \geqq T(F_{pq}(x), F_{qr}(y))$$

is satisfied. Then $\mathscr{U}$ *is the basis for a Hausdorff uniformity on* $S \times S$.

Proof. We verify that the $U(\varepsilon, \lambda)$ satisfy the axioms for a basis for a Hausdorff (or separated) uniformity as given in [2; p. 174–180] (or in [1; II, § 1, n°1]).

(a) Let $\Delta = \{(p, p); p \in S\}$ and $U(\varepsilon, \lambda)$ be given. Since for any $p \in S$, $F_{pp}(\varepsilon) = 1$, it follows that $(p, p) \in U(\varepsilon, \lambda)$. Thus $\Delta \subset U(\varepsilon, \lambda)$.

(b) Since $F_{pq} = F_{qp}$, $U(\varepsilon, \lambda)$ is symmetric.

(c) Let $U(\varepsilon, \lambda)$ be given. We have to show that there is a $W \in \mathscr{U}$ such that $W \circ W \subset U$. To this end, choose $\varepsilon' = \varepsilon/2$ and λ' so small that $T(1 - \lambda', 1 - \lambda') > 1 - \lambda$. Suppose now that (p, q) and (q, r) belong to

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[1] These considerations have led to the study of *SM* spaces which are not metrizable as well as to other natural topologies for *SM* spaces, questions which will be investigated in a subsequent paper.

[2] The terminology and notation are as in [3].