ON NORMAL NUMBERS

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1. Introduction. A real number ξ, $0 \leqq \xi < 1$, is said to the *normal in the scale of* r (or *to base* r), if in $\xi = 0 \cdot a_1 a_2 \cdots$ expanded in the scale of r[1] every combination of digits occurs with the proper frequency. If $b_1 b_2 \cdots b_k$ is any combination of digits, and Z_N the number of indices i in $1 \leqq i \leqq N$ having

$$b_1 = a_i, \cdots, b_k = a_{i+k-1} ,$$

then the condition is that

$$\lim_{N \to \infty} Z_N N^{-1} = r^{-k} . \tag{1}$$

A number ξ is called *simply normal* in the scale of r if (1) holds for $k = 1$. A number is said to be *absolutely normal* if it is normal to every base r. It is well-known (see, for example, [6], Theorem 8.11) that almost every number ξ is absolutely normal.

We write $r \sim s$, if there exist integers n, m with $r^n = s^m$. Otherwise, we put $r \nsim s$.

In this paper we solve the following problem. *Under what conditions on r, s is every number ξ which is normal to base r also normal to base s?* The answer is given by

THEOREM 1. *A Assume $r \sim s$. Then any number normal to base r is normal to base s.*

B If $r \nsim s$, then the set of numbers ξ which are normal to base r but not even simply normal to base s has the power of the continuum.

The A-part of the Theorem is rather trivial, but I shall sketch a proof of it, since I could not find one in the literature.

Next, let I be an interval of length $|I|$ contained in the unit-interval $U = [0, 1]$. We write $M_N(\xi, r, I)$ for the number of indices i in $1 \leqq i \leqq N$ such that the fractional part $\{r^i \xi\}$ of $r^i \xi$ lies I. A sequence $\xi, r\xi, r^2\xi, \cdots$ *has uniform distribution modulo* 1 if

$$R_N(\xi, r, I) = M_N(\xi, r, I) - N|I| = o(N)$$

for any I. It was proved by Wall [8] (the most accessible proof in [6], Theorem 8.15) that ξ is normal to base r if and only if $\xi, r\xi, r^2\xi, \cdots$ has uniform distribution modulo 1.

Write $T_{s,t}$, where $1 < t < s$, for the following mapping in U: If $\xi = 0 \cdot a_1 a_2 \cdots$ in the scale of t, then $T_{s,t}\xi = 0 \cdot a_1 a_2 \cdots$ in the scale of s.

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[1] In case of ambiguity we take the representation with an infinity of a_i less then $r - 1$. But this does not affect the property of ξ to be normal or not.