ON THE EXTENSIONS OF A TORSION MODULE

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This paper concerns the structure of $\operatorname{Ext}(A, T) = \operatorname{Ext}^1_R(A, T)$ where A is a torsion-free and T is a torsion module over a Dedekind ring R. In the first section it is shown that for a given torsion-free module A the structure of $\operatorname{Ext}(A, T)$ is completely determined by the basic subgroup of T. If in addition T is primary the structure of $\operatorname{Ext}(A, T)$ depends on a single known invariant of T, called by Szele [4] the critical number. The rest of the paper is devoted to showing the nature of this dependence in the special case in which A is the quotient field of R and T is primary. The problem reduces to that of computing the rank of certain complete modules over a discrete valuation ring. This computation is carried out in section two and the description of $\operatorname{Ext}(A, T)$ is given in section three.

Throughout the paper R is assumed to be a Dedekind ring other than a field. A consequence of this assumption, used in section two, is that R is infinite. An exact sequence $0 \to A' \to A \to A'' \to 0$ and a module C give rise to two exact sequences. We follow S. MacLane in calling the one beginning $0 \to \operatorname{Hom}(A'', C)$ the *first exact sequence* and the one beginning $0 \to \operatorname{Hom}(C, A')$ the *second exact sequence.*

1. In this section it is shown that whenever A is torsion-free and C is a torsion module, then the structure of $\operatorname{Ext}(A, C)$ depends only on the basic submodule of C.

LEMMA 1.1. *If A, B, C are modules with A torsion-free and if there is a homomorphism of B into C with divisible cokernel, then* $\operatorname{Ext}(A, C)$ *is a direct summand of* $\operatorname{Ext}(A, B)$.

Proof. Suppose that $f : B \to C$ is a homomorphism with $\operatorname{Coker} f = C/Im f$ divisible. Let f be factored into an epimorphism g followed by a monomorphism $h : f = hg$. We get two exact sequences

$$0 \longrightarrow Im f \xrightarrow{h} C \longrightarrow \operatorname{Coker} f \longrightarrow 0$$
$$0 \longrightarrow \operatorname{Ker} f \longrightarrow B \xrightarrow{g} Im\ f \longrightarrow 0\,,$$

and the relevant parts of the associated second exact sequences are

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