ON NON-ASSOCIATIVE ALGEBRAS ASSOCIATED WITH BILINEAR FORMS

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If $\mathfrak{V}_0$ is a vector space over a field k, then with any non-degenerate bilinear form f_0 on $\mathfrak{V}_0 \times \mathfrak{V}_0$ is associated the group $\mathfrak{G}$ of linear transformations of $\mathfrak{V}_0$ which keep f_0 invariant. In this paper a procedure is given for associating with such a bilinear form an algebra $\mathfrak{A}$, non-associative in general, whose automorphism group is isomorphic to $\mathfrak{G}$ and which is right and left simple provided $\mathfrak{V}_0$ has dimension at least 2. In case k is the field of real numbers, then $\mathfrak{G}$ is a Lie group and its Lie algebra is the Lie algebra of derivations of $\mathfrak{A}$. In case the form f_0 is degenerate, and either symmetric or alternating, then the analogue of the Wedderburn Principal Theorem holds for $\mathfrak{A}$. The results obtained apply, in particular, to the orthogonal and symplectic groups.

Let $\mathfrak{V}_0$ be a vector space of dimension n over a field k with basis $u_1, \cdots, u_n$. It is assumed that $\lambda v = v\lambda$ for all $v \in \mathfrak{V}_0$ and $\lambda \in k$. Suppose f_0 is a bilinear form on $\mathfrak{V}_0 \times \mathfrak{V}_0$. Define $\mathfrak{A}$ to be the algebra over k with basis $e_0, e_1, \cdots, e_n$ and multiplication table $e_0^2 = e_0$, $e_i \cdot e_0 = e_0 \cdot e_i = e_i$, $e_i \cdot e_j = f(e_i\, e_j)e_0$ for $i, j = 1, 2, \cdots, n$, where $f(e_i, e_j) = f_0(u_i, u_j)$. Let $\mathfrak{V}$ be the subspace of $\mathfrak{A}$ spanned be $e_1, \cdots, e_n$. Then f is a bilinear form on $\mathfrak{V} \times \mathfrak{V}$.

THEOREM 1. *Suppose that f is non-degenerate and that $n \geqq 2$. Then $\mathfrak{A}$ is right and left simple.*

Proof. Let $\mathfrak{U}$ be a non-zero left ideal of $\mathfrak{A}$ and let u be a non-zero element of $\mathfrak{U}$. Suppose first that $u \in \mathfrak{V}$. Then there exists an element $v \in V$ such that $f(v, u) \neq 0$. Then $v \cdot u = f(v, u)e_0$. Therefore $e_0 \in \mathfrak{U}$ and so $\mathfrak{U} = \mathfrak{A}$. Next suppose $u = \alpha e_0 + v$ where $\alpha \neq 0$ in k and $v \in V$. Then one can assume $\alpha = 1$. Since $n \geqq 2$ it follows that $e_1 \cdot u = e_1 + \lambda_1 e_0$ and $e_2 \cdot u = e_2 + \lambda_2 e_0$ where $\lambda_1, \lambda_2 \in k$. If $\lambda_1 = 0$ then $e_1 \in U$ and the first part of the proof applies; similarly if $\lambda_2 = 0$. Consequently one can suppose $\lambda_1\lambda_2 \neq 0$. Then $\lambda_2 e_1 u - \lambda_1 e_2 u = \lambda_2 e_1 - \lambda_1 e_2$ is a non-zero element in $\mathfrak{U} \cap \mathfrak{V}$. Thus the first part of the proof again applies and so $\mathfrak{U} = \mathfrak{A}$. Therefore $\mathfrak{A}$ is left simple; similarly $\mathfrak{A}$ is right simple.

If $\mathfrak{A}$ is any (non-associative) algebra over k then left (right) multiplication by an element $a \in \mathfrak{A}$ determines a linear transformation $L_a(R_a)$ of the underlying vector space of $\mathfrak{A}$ by $a \cdot u = L_a u (u \cdot a = R_a u)$, $u \in \mathfrak{A}$. The set of linear transformations L_a (R_a) for $a \in \mathfrak{A}$ generate an associative algebra $L(\mathfrak{A})$ $(R(\mathfrak{A}))$ over k. The algebras $L(\mathfrak{A})$ and $R(\mathfrak{A})$ together

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