SEPARABLE CONJUGATE SPACES

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A Banach space B is reflexive if the natural isometric mapping of B into the second conjugate space B^{**} covers all of B^{**}. All conjugate spaces of a reflexive separable space B are separable. The nonreflexive space $l^{(1)}$ is separable and its first conjugate space is (m), which is nonseparable. The space (c_0) is separable, its first conjugate space is $l^{(1)}$, and its second conjugate space is (m). An example is known of a nonreflexive Banach space whose conjugate spaces are all separable [4]. This space is pseudo-reflexive in the sense that its natural image in the second conjugate space has a finite-dimensional complement. The structure of such spaces has been studied carefully [2].

The main purpose of this paper is to show that the sequence started by $l^{(1)}$ and (c_0) can be extended to give a sequence $\{B_n\}$ of separable Banach spaces such that, for each n, the nth conjugate space of B_n is its first nonseparable conjugate space. The principal tool used is a theorem which states a sufficient condition on a space T for the existence of a space B with

$$B^{**} = \pi(B) \dotplus T ,$$

where $\pi(B)$ is the natural image of B in B^{**}. The following definition and notation will be used.

A *basis* for a Banach space B is a sequence $\{u^i\}$ such that, for each x of B, there is a unique sequence of numbers $\{a_i\}$ for which $\lim_{n\to\infty} \| x - \sum_1^n a_i u_i \| = 0$. A sequence $\{u_i\}$ is a basis for its closed linear span if and only if there is a number $\varepsilon > 0$ such that

$$\left\|\sum_1^{n+p} c_i x_i\right\| \geq \varepsilon \left\|\sum_1^{n} c_i x_i\right\|$$

for any numbers $\{c_i\}$ and positive integers n and p [1, page 111]. If ε can be $+1$, the basis is an *orthogonal basis*. It will be useful to classify bases as follows:

Type α. If $\{a_i\}$ is a sequence of numbers for which $\sup_n \| \sum_1^n a_i u_i \| < \infty$, then $\sum_1^\infty a_i u_i$ converges.

Type β. If f is a linear functional defined on B and $\|f\|_n$ is the norm of f on the closed linear span of $\{u_i \mid i \geq n\}$, then $\lim_{n\to\infty} \|f\|_n = 0$.

There are Banach spaces which have bases which are neither of type α nor of type β, while a basis is of both types if and only if the space

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