MEASURES DEFINED BY ABSTRACT L_p SPACES

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Let a linear space L of real-valued functions on a set E and a semi-norm on L be given. We shall consider when there exists a countably additive measure on E such that L is L_p with respect to this measure. We shall prove that certain conditions are sufficient for the measure to exist; it is obvious that these conditions are necessary. (We consider only the case where the constant function $1 \in L$.)

We need not assume that the elements of L are functions on a set. If we do not make this assumption, we use a theorem of Kakutani ([3], p. 998) to construct a representation for L as a space of continuous functions on a compact Hausdorff space. If, however, the elements of L are given as functions, we leave this preëstablished representation unchanged, even when it is not the one given by Kakutani's theorem.

The case where $p = 1$ and the elements of L are not given as functions was treated by Kakutani [2]. The case $p = 2$ will receive special attention at the end of the present paper. In this latter case, one may replace some of the hypotheses of the general case by the hypothesis that the semi-norm on L arises from a positive semi-definite bilinear form.

Let L be a Riesz space whose elements are functions on a set E. That is, let L be a set of real-valued functions on E which contains with f, g:

(a) $f + g$ defined by $(f + g)(x) = f(x) + g(x)$,

(b) αf defined by $(\alpha f)(x) = \alpha[f(x)]$, for each real number α,

(c) $f \wedge g$ defined by $(f \wedge g)(x) = \min(f(x), g(x))$,

and (d) $f \vee g$ defined by $(f \vee g)(x) = \max(f(x), g(x))$.

We denote $f \vee 0$ by f^+ and $(-f) \vee 0$ by f^-. (The case where L is an abstract Banach lattice will be considered shortly.)

Let p be a fixed real number ≥ 1. Throughout the paper, p will always stand for this fixed number. We suppose there is a semi-norm, which we denote by $\|\ \ \|$, defined on L. We further suppose:

(1) L is complete. That is, if $f_1, f_2, \cdots \in L$ are such that $\|f_n - f_m\|$ is small for large n, m; then there is a $g \in L$ such that $\|g - f_n\| \to 0$.

(2) For each $f \in L$, $\| |f| \| = \|f\|$.

(3) If f, g are positive, $\|f + g\|^p \geq \|f\|^p + \|g\|^p$.

(4) If f, g are positive and $f \wedge g = 0$, $\|f + g\|^p \leq \|f\|^p + \|g\|^p$.

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