COMPUTATIONS OF THE MULTIPLICITY FUNCTION

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1. Introduction. Let H be a separable Hilbert space. The following two problems will be studied:

1. Given a bounded normal operator A, of multiplicity m, what are the conditions, on the bounded measurable function f, so that the multiplicity of $S = f(A)$ is $n, n < \infty$?

2. How to compute the multiplicity of a normal operator that commutes with a given normal operator, of finite multiplicity?

NOTATION. Let S be a normal operator of multiplicity n, $n < \infty$. There exist a Borel measure μ and n Borel sets in the complex plane $e_1 \supset e_2 \supset \cdots \supset e_n$, such that, up to unitary equivalence,

$$H = \sum_{i=1}^{n} L_2(\mu, e_i) \tag{1.1}$$

$$S\begin{pmatrix} f_1(\lambda) \\ \vdots \\ f_n(\lambda) \end{pmatrix} = \begin{pmatrix} \lambda f_1(\lambda) \\ \vdots \\ \lambda f_n(\lambda) \end{pmatrix}$$

This is the Multiplicity Theorem. (See Theorem X. 5.10) of [1]. The operator S has uniform multiplicity if $e_1 = e_2 = \cdots = e_n$.

The resolution of the identity, of a normal operator A, will be denoted by $E(A; \alpha)$. The Boolean algebra of projections, generated by $E(A; \alpha)$ will be denoted by $\mathfrak{E}_A$. Let $E(\alpha)$ stand for $E(S; \alpha)$ and $\mathfrak{E}$ for $\mathfrak{E}_S$. Throughout this note all operators are assumed to be bounded.

We shall use the following results from [2]:

Let S be a normal operator of multiplicity n, and B a normal operator that commutes with S. Let H and S be represented by 1.1.

THEOREM A. *There exist k Borel measurable bounded complex functions $y_1(\lambda), \cdots, y_k(\lambda)$ and k matrices of Borel measurable bounded complex functions $\varepsilon_1(\lambda), \cdots, \varepsilon_k(\lambda)$ such that:*

For a fixed λ the matrices $\varepsilon_i(\lambda)$ are disjoint self adjoint projections whose sum is the identity and

$$B\begin{pmatrix} f_1(\lambda) \\ \vdots \\ f_n(\lambda) \end{pmatrix} = \left(\sum_{i=1}^{k} y_i \varepsilon^i(\lambda)\right)\begin{pmatrix} f_1(\lambda) \\ \vdots \\ f_n(\lambda) \end{pmatrix}. \tag{1.2}$$

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