UNIONS OF CELL PAIRS IN E^3

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In [4] it is shown that there are pairs of cells of all dimensions possible in euclidean 3-space, E^3, which are tame separately, but which have a wild set as their union. Such pairs can be constructed when the individual cells intersect in a single point. The present paper gives conditions that unions of some such pairs be tame sets as well as a number of other results.

LEMMA 1. *Let D_1 be a disk which is polyhedral and which lies on the boundary, ∂T, of a tetrahedron T in E^3. If D_2 is a disk in E^3 which has a polygonal boundary and is locally polyhedral mod ∂D_2 while $D_2 \cap T = D_2 \cap D_1 = \partial D_2 \cap \partial D_1 = J$, an arc, then $D_1 \cup D_2$ is a tame disk.*

Proof. Let P_1 and P_2 be polyhedral disks in ∂T, $P_1 \cap P_2 = \square$ and $(P_1 \cup P_2) \cap D_1 = \square$. Then $\overline{\partial T \backslash (P_1 \cup P_2)}$ is a polyhedral annulus, A_1. If Q is a polyhedral disk in $D_2 \backslash \partial D_2$, then $\overline{D_2 \backslash Q}$ is an annulus A_2 which is locally polyhedral mod ∂D_2. By applying Lemma 5.1 of [8] to A_1 and A_2 one obtains a space homeomorphism h carrying E^3 onto E^3 while $h(D_1 \cup D_2)$ is a polyhedral set. This completes the proof of Lemma 1.

LEMMA 2. *Let D_1 be the disk of Lemma 1 while D_2 is a tame disk in E^3 such that $D_2 \cap T = D_2 \cap D_1 = \partial D_2 \cap \partial D_1 = J$, an arc. Then $\partial T \cup \partial D_2$ is tame.*

Proof. By Theorem 2 of [3] $\partial D_1 \cup \partial D_2$ is locally tame and hence tame by [1] or [8]. Let a be a point of ∂J and J' be an interval of ∂D_1 having a as an end point and $J' \cap \partial D_2 = a$. We choose a polygonal disk M on ∂T with $(J'/\partial J')$ in its interior while $\partial D_1 \cap M = J'$. By a swelling [5] of M toward the component of $E^3 \backslash \partial T$ which meets ∂D_2 we obtain a disk M' which is locally polyhedral mod ∂M and $M' \cap \partial T = \partial M = \partial M'$. The sphere $S = M' \cup (\partial T \backslash M)$ is tame by [8] and S is pierced at a by a tame arc lying on $\partial(D_1 \cup D_2)$. Hence by [7] $\partial D_2 \cup S$ is locally tame at a. We select an arc P in $(S \backslash M') \cup a$ which is locally polyhedral except at the point a. There is an arc A on ∂D_2 which lies in the exterior of S except for its end point a. The arc $A \cup P$ is tame since $S \cup \partial D_2$ is tame. Let the arc P be swollen into a 3-cell C^3 with P in its interior such that C^3 is locally polyhedral mod a, $C^3 \cap S$ is a disk while $C^3 \cap M = a$. Then ∂C^3 is pierced at a by $A \cup P$ and so $A \cup P \cup \partial C^3$ is tame by [7]. Evidently there is an arc P' on ∂C^3 so

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