BOUNDS FOR THE EIGENVALUES OF SOME VIBRATING SYSTEMS

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1. Introduction. If a string with a non-negative integrable density $\rho(x)$, $x \in [a, b]$, is fixed at the points $x = a$ and $x = b$ under unit tension, then the natural frequencies of the string are determined by the eigenvalues of the boundary value problem

$$y'' + \mu\rho(x)y = 0 , \quad y(a) = y(b) = 0 . \tag{1.1}$$

Indicating their dependence on the function $\rho(x)$, we denote these eigenvalues by

$$\mu_1[\rho] < \mu_2[\rho] < \cdots . \tag{1.2}$$

We consider the set of all such strings which have the same total mass, $M = \int_a^b \rho(x)dx$. It is well known [5] that the eigenvalues (1.2) satisfy the inequality

$$\mu_n[\rho] \geq \frac{4n^2}{M(b-a)} , \qquad n = 1, 2, \cdots , \tag{1.3}$$

with equality when a mass of amount M/n is concentrated at the midpoint of each of n segments obtained by partitioning the string into n equal parts. If we place some restriction on $\rho(x)$ which prohibits such an accumulation of mass, then we can expect to get a larger bound than that of (1.3). M. G. Krein [8] has found that when $0 \leq \rho(x) \leq H < \infty$, the eigenvalues (1.2) satisfy the inequalities

$$\frac{4Hn^2}{M^2} X\left(\frac{M}{H(b-a)}\right) \leq \mu_n[\rho] \leq \frac{Hn^2\pi^2}{M^2} , \tag{1.4}$$

where $X(t)$ is the least positive root of the equation

$$\sqrt{X} \tan X = \frac{t}{1-t} .$$

The inequality (1.4) is sharp and as $H \to \infty$, the lower bound approaches that of (1.3).

In this paper, we investigate the nature of the density functions for which the greatest lower bounds of the eigenvalues (1.2) are attained

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