FRACTIONAL POWERS OF CLOSED OPERATORS AND THE SEMIGROUPS GENERATED BY THEM

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Fractional powers of closed linear operators were first constructed by Bochner [2] and subsequently Feller [3], for the Laplacian operator. These constructions depend in an essential way on the fact that the Laplacian generates a semigroup. Phillips [6] in fact showed that these constructions (for positive indices less than one) were part of a more general one based on the Kolmogoroff-Levy representation theorem for infinitely divisible distributions. Finally, the present author constructed an operational calculus [1] for infinitesimal generators affording in particular a systematic study of the representation and properties of these operators.

In this paper we obtain a new construction for fractional powers in which it is not required that the base operator generate a semigroup; indeed its domain need not even be dense. Since the semigroup is not available, the previous constructions, based as they are on the Riemann-Liouville integrals, are not possible. However, we shall show, if the resolvent exists for $\lambda > 0$, and is $O(1/\lambda)$ for all λ, (a weaker condition will suffice at the origin, see § 7), then fractional powers may still be constructed, using an abstract version of the Stieltjes transform.

It is immediate that a closed operator A, for which $\|\lambda R(\lambda, A)\| < M$, does not necessarily generate a semigroup of any type. For a simple example, let the Banach space be $l_2(-\infty, \infty)$ and let A correspond to multiplying the nth coordinate by $n(1+i)$ say. Then for $\lambda > 0$, $\|R(\lambda, A)\| \leq \sqrt{2}/\lambda$, whereas A does not generate a semigroup, since no right-half plane is free of the spectra of A. An example in which A has no spectra in the right half plane and yet no semigroup is generated is given by Phillips [4].

The main motivation for the construction of fractional powers is the application to abstract Cauchy problems of the type:

$$\frac{d^n}{dt^n}u(t) \pm Au(t) = 0 \tag{1}$$

for $n \geq 2$, and it turns out that for the solution of (1.1), A itself need not be an infinitesimal generator. In this paper we study only the case $n = 2$, and we expect to consider the general case later.

The properties of newly constructed fractional powers are identical with those obtained in [1] for the case where A is a generator, with one important difference; namely that $-(-A)^\alpha$ generates semigroups in

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