THE PALEY-WIENER THEOREM IN METRIC LINEAR SPACES

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1. Introduction. By a *basis* in a topological linear space $\mathscr{T}$ we mean a sequence $\{x_n\}$ of points of $\mathscr{T}$ such that to every x in $\mathscr{T}$ there corresponds a unique sequence $\{a_n\}$ of scalars for which

$$x = \sum_{n=1}^{\infty} a_n x_n \ .$$

Denoting the coefficient functionals here by φ_n, we can rewrite this as

$$x = \sum_{n=1}^{\infty} \varphi_n(x) x_n \ . \tag{1.1}$$

If it happens that all φ_n are continuous on $\mathscr{T}$, the basis will be referred to as a *Schauder basis*. Every basis in a Fréchet space [**14**, pp. 59, 110] is known to be a Schauder basis (see Newns [**21**], pp. 431–432), and it will be shown here that the same holds for bases in an arbitrary complete metric linear space over the real or complex field.

The classical Paley-Wiener theorem asserts that for $\mathscr{T}$ a Banach space, all sequences which sufficiently closely approximate bases must themselves be bases. A more precise statement of the theorem is obtained by replacing $\mathscr{M}$ in Theorem 1 by a Banach space $\mathscr{B}$.

The bibliography at the end of the present paper includes a chronological listing of articles on the Paley-Wiener theorem, and we give now a brief résumé of its history. As originally presented in 1934 by Paley and Wiener [**1**, p. 100], the theorem was derived specifically for the Hilbert space L^2. Then, in applying the theorem to the Pincherle basis problem [**2**, p. 469], Boas observed in 1940 that the proof of Paley and Wiener remains valid for Banach spaces. Boas also succeeded in simplifying a portion of the proof. However, the first really elementary proof of the theorem was published in 1949 by Schafke [**8**], to whom conclusion (3) is due. The remaining articles on the Paley-Wiener theorem deal mainly with various generalizations of condition (2.1) for Hilbert spaces.

From the viewpoint of modern functional analysis, the key to theorems of Paley-Wiener type lies in the inversion of an operator $I+T$ by means of a geometric series in T. This crucial observation was made by Buck [**15**, p. 410] in 1953.[1]

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[1] The same technique was used also in [**9**], the author having been unaware of the earlier remarks of Buck. A further application (to generalized bases) appears in [**12**].