PERMANENTS OF CYCLIC MATRICES

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1. Introduction. Let $A=[a_{ij}]$ be an $n\times n$ matrix with non-negative real entries. The *permanent* of A, written $P(A)$, is defined by

$$P(A) = \sum a_{1i_1}a_{2i_2}\cdots a_{ni_n}\ , \tag{1.1}$$

where the summation extends over the $n!$ permutations of the integers $i_1, i_2, \cdots, i_n$. Thus the permanent and determinant are alike in definition except for sign changes. However unlike the determinant, the properties of the permanent function are little understood. The object of this paper is to determine for a certain class of matrices those matrices A for which the permanent and determinant are equal in absolute value. This property we write $P(A) = |D(A)|$. For such matrices the permanent may then be evaluated by the determinant.

Let $A = [a_{ij}]$ be an $n \times n$ matrix composed of 0's and 1's with row and column sums equal to s. Let $\Sigma = [\sigma_{ij}]$ be a permutation submatrix of A. This means that Σ is a permutation matrix of order n such that $\sigma_{kl} = 1$ implies $a_{kl} = 1$. With Σ we associate a permutation Σ' of the letters $1, 2, \cdots, n$

$$\Sigma'(i) = j \text{ if and only if } \sigma_{ij} = 1\ . \tag{1.2}$$

It follows by definition then that $P(A) = |D(A)|$ if and only if every Σ' is even or else every Σ' is odd.

By a theorem due to König (1), the matrix A may be written as a sum of s permutation matrices,

$$A = \pi_1 + \pi_2 + \cdots + \pi_s\ . \tag{1.3}$$

For convenience we will say that A is *defined* by the s permutations $\pi'_1, \pi'_2, \cdots, \pi'_s$. If $\pi'_k\pi'_j = \pi'_j\pi'_k$ for each j and k, then A will be called *abelian*. If for $i = 1, 2, \cdots, s$, $\pi'_i = (1, 2, \cdots, n)^{d_i}$ where $0 \leqq d_i < n$, then A is cyclic and will be said to be defined by the difference $d_1, d_2, \cdots, d_s$ mod n.

Now let C be the 7×7 cyclic matrix defined by the differences $0, 1, 3$, mod 7. The main result of the paper may be stated as follows:

Let A be an $n \times n$ abelian matrix with $s \geqq 3$ ones in each row and column. Then $P(A) = |D(A)|$ if and only if $s = 3$, $n = 7e$ and upon permutations of rows and columns A is transformed into the direct sum of C taken e times.

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