MINIMUM PROBLEMS IN THE THEORY OF PSEUDO-CONFORMAL TRANSFORMATIONS AND THEIR APPLICATION TO ESTIMATION OF THE CURVATURE OF THE INVARIANT METRIC

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1. Introduction. Resume of some previous results.[1] Let B be a domain in the z_1, z_2-space[2] possessing a Bergman kernel function $K^{(B)}(z_1, z_2; \bar{t}_1, \bar{t}_2)$, $(z_1, z_2) \in B$, $(t_1, t_2) \in B$. By identifying the arguments $(t_1, t_2) = (z_1, z_2)$ one obtains the function $K^{(B)} \equiv K^{(B)}(z_1, z_2) \equiv K^{(B)}(z_1, z_2; \bar{z}_1, \bar{z}_2)$ which plays an essential role in the theory of pseudo-conformal transformations. An important application to this theory is the theorem proved by S. Bergman stating that the metric

$$ds^2 = \sum_{m,n=1}^{2} T^{(B)}_{m\bar{n}} dz_m d\bar{z}_n, \quad T_{m\bar{n}} = \frac{\partial^2 \log K^{(B)}}{\partial z_m \partial \bar{z}_n} \tag{1.1}$$

is invariant under pseudo-conformal transformations (B. [1], p. 52). From this follows that all measures of geometric objects in B which are based on the metric (1.1) are also invariant under pseudo-conformal transformations.

In the present paper we are concerned in particular with the *Riemann Curvature of* (1.1) *in an analytic direction* (see definition in section 3). Since the second derivatives of the function $\log K^{(B)}(z_1, z_2)$ are the main constituent in the definition of the curvature, we at first discuss bounds for their distortion under pseudo-conformal transformation (see Theorem 1). For this purpose, *Bergman's method of the minimum integral* is used (B. [3], p. 48; K. [1]; S. [1]):

Relations among various solutions of minimum problems of the type

$$\int_B |f(z)|^2 d\omega \equiv \min = \lambda_B \quad (d\omega = \text{volume element}), \tag{1.2}$$

are studied (see Theorems 2 and 3). Here $f(z)$ are analytic functions,

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[1] Square brackets refer to the bibliography at the end of the paper. We use the abbreviations B. = Bergman, F. = Fuchs, K. = Kobayashi, S. = Stark.

[2] In the present paper we consider only domains in the space of two complex variables. The generalization of the methods to the space of more complex variables involves difficulties of technical nature only.