THE ABSOLUTE CONTINUITY OF TOEPLITZ'S MATRICES

MARVIN ROSENBLUM

1. Introduction. Suppose W is a real $L^2(-\pi, \pi)$ function that is bounded below but not equivalent to a constant function. The *Toeplitz matrix* associated with W is $\boldsymbol{T}_0 = [w_{j-k}]$, $j, k = 0, 1, 2, \cdots$, where

$$(1.1) \qquad w_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} W(\phi)e^{-in\phi}\, d\phi,\ n = 0, \pm 1, \pm 2, \cdots$$

The hermitian matrix $\boldsymbol{T}_0$ gives rise to a semi-bounded transformation $\boldsymbol{T}_1$ on complex sequential Hilbert space l_2, and thus the Friedrichs extension $\boldsymbol{T}$ of $\boldsymbol{T}_1$ is a self-adjoint operator. $\boldsymbol{T} = \boldsymbol{T}(W(\phi))$ is the *Toeplitz operator* associated with W.

In [5], [6] Hartman and Wintner show that the case in which W is not semi-bounded (which we prudently avoid here) presents special difficulty. However for semi-bounded W they prove that

(i) the spectrum of $\boldsymbol{T}$ fills the interval
[ess inf W, ess sup W],
and

(ii) $\boldsymbol{T}$ has no point spectrum.

Thus the spectral measure ([4], p. 58) $\boldsymbol{E}(\cdot)$ of $\boldsymbol{T}$ is such that $\langle \boldsymbol{E}(\cdot)F, F\rangle$ is a nonatomic Borel measure for each $F \in l^2$. If $\langle \boldsymbol{E}(\cdot)F, F\rangle$ is AC (absolutely continuous with respect to Lebesgue measure) for each $F \in l^2$, then we say that $\boldsymbol{T}$ is AC.

Our investigation continues work of C. R. Putnam [11]. He proves that $\boldsymbol{T}$ is AC in each of the following cases:

(i) $W(\phi) = 2\cos n\phi$, $n = 1, 2, \cdots$

(ii) $W(\phi) = 2\sin n\phi$, $n = 1, 2, \cdots$

(iii) Let $a_{jk} = w_{k-j}$ for $k - j \geqq 1$ and $a_{jk} = 0$ otherwise.

Further suppose that the $\{w_n\}$ are real, that $\boldsymbol{A}_0 = [a_{jk}]$ is bounded, and that 0 is not an eigenvalue of the Hankel matrix $[w_{j+k+1}]$, $j, k = 0, 1, 2, \cdots$.

For case (i) Putnam gives a more complete spectral analysis. He applies the perturbation theory propounded in [13] to prove the following result:

1.2 $\boldsymbol{T}(2\cos n\phi)$ is unitarily equivalent to $2T_n(\frac{1}{2}\boldsymbol{T}(2\cos\phi))$. Here T_n is the nth degree Tchebichef polynomial, $n = 1, 2, \cdots$.

Received July 20, 1959. This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under contract AF 49(638)-72 at the University of Virginia.