A REFINEMENT OF THE FUNDAMENTAL THEOREM ON THE DENSITY OF THE SUM OF TWO SETS OF INTEGERS

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Let $A = \{a_0 < a_1 < \cdots\}$ be a set of integers and let $A(n)$ be the number of integers in A not exceeding n. If A, B are two such sets, we put $A + B = \{a + b\}$, where a denotes generically an element of A, b an element of B. It should be noted that A and B may contain negative numbers or zero and that these are counted in $A(n)$ and $B(n)$.

Erdoes in an unpublished paper proved:

If $\lim_{m\to\infty}(A(m)/m) = \lim_{m\to\infty}(B(m)/m) = 0$, then for every $\varepsilon > 0$ there are infinitely many x such that if $C = A + B$ then

$$C(x) \geq A(x)(1 - \varepsilon) + B(x) .$$

Clearly there are then also infinitely many y such that

$$C(y) \geq A(y) + B(y)(1 - \varepsilon) .$$

Erdoes conjectured that it is possible to choose infinitely many $x = y$.

At the Number Theory Conference in Boulder, Colorado, Erdoes proposed this problem to the author. It is clear that the Fundamental Theorem [3] is inadequate to deal with this problem, because it fails if $1 \notin C$. The search for a stronger theorem finally led the author to Theorem 2. Theorem 3 is a consequence of Theorem 2 and is considerably stronger than Erdoes conjecture.

THEOREM 1. *Let* $a_0 = b_0 = 0$. If $n \geq 0, n \notin C$ *then there is an* $m \notin C$, $m = n$ *or* $m < (n/2)$, *such that*

(1)
$$\frac{C(n)}{n+1} \geq \frac{A(m) + B(m) - 1}{m+1} + \left(C(n - m - 1) - \frac{C(n)}{n+1}(n - m)\right)\frac{1}{m+1} .$$

For the proof of Theorem 1, we consider the following transformation: Let $n_1 < n_2 < \cdots < n_r = n$ be the gaps in C. Form $d_i = n - n_i$. Choose, if possible, a fixed number $e \in B$ such that an equation

$$a + e + d_i = n_j \tag{2}$$

holds for some i. Let the set B' consist of all numbers $e + d_s$ for which

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