A CHARACTERISTIC SUBGROUP OF A p-GROUP

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If x, y are elements and H, K subsets of the p-group G, we shall denote by $[x, y]$ the element $y^{-p}x^{-p}(xy)^p$ of G, and by $[H, K]$ the subgroup of G generated by the set of all $[h, k]$ for h in H and k in K. We call a p-group G *p-abelian* if $(xy)^p = x^p y^p$ for all elements x, y of G. If we let $\theta(G) = [G, G]$ then $\theta(G)$ is a characteristic subgroup of G and $G/\theta(G)$ is p-abelian. In fact, $\theta(G)$ is the minimal normal subgroup N of G for which G/N is p-abelian. It is clear that $\theta(G)$ is contained in the derived group of G, and $G/\theta(G)$ is *regular* in the sense of P. Hall [3]

Theorem 1 lists some elementary properties of p-abelian groups. These properties are used to obtain a characterization of p-groups G (for $p \geq 3$) in which the subgroup generated by the pth powers of elements of G coincides with the Frattini subgroup of G (Theorems 2 and 3). A group G is said to be metacyclic if there exists a cyclic normal subgroup N with G/N cyclic. Theorem 4 states that a p-group G, for $p > 2$, is metacyclic if and only if $G/\theta(G)$ is metacyclic. Theorems on metacyclic p-groups due to Blackburn and Huppert are obtained as corollaries of Theorems 3 and 4.

The following notation is used: G is a p-group; $G^{(n)}$ is the nth derived group of G; G_n is the nth element in the descending central series of G; $P(G)$ is the subgroup of G generated by the set of all x^p for x belonging to G; $\Phi(G)$ is the Frattini subgroup of G; $\langle x, y, \cdots \rangle$ is the subgroup generated by the elements $x, y, \cdots$; $Z(G)$ is the center of G; $(h, k) = h^{-1}k^{-1}hk$; if H, K are subsets of G, then (H, K) is the subgroup generated by the set of all (h, k) for $h \in H$ and $k \in K$.

THEOREM 1. *If G is p-abelian, then*

$$P(G^{(1)}) = P(G)^{(1)}\ , \tag{1.1}$$

$$P(G) \subseteq Z(G)\ , \tag{1.2}$$

$$\Phi(G^{(1)}) = \Phi(G)^{(1)} = G^{(2)}\ . \tag{1.3}$$

Proof of (1.1). $\theta(G) = \langle 1 \rangle$ implies that $(xyx^{-1}y^{-1})^p = x^p y^p x^{-p} y^{-p}$ for all x, y in G. (1.1) follows immediately.

Proof of (1.2). Let x be an arbitrary element of G, and suppose the order of x is p^n. Let $u = x^{1+p+\cdots+p^{n-1}}$. Then, for any y in G,

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