LIE MAPPINGS IN CHARACTERISTIC 2

I. N. HERSTEIN AND ERWIN KLEINFELD[1]

1. Introduction. In a previous paper [2] one of the authors proved that Jordan homomorphism, that is, an additive mapping φ onto a prime ring of characteristic not 2 or 3 which preserves squares, is either a homomorphism or an anti-homomorphism. Smiley [6] then showed that this was also true in the characteristic 3 case; in the characteristic 2 case he showed that the same conclusion holds for φ if one assumes $\varphi(aba) = \varphi(a)\varphi(b)\varphi(a)$ for all a and b.

We concern ourselves here with mappings φ onto a simple ring of characteristic 2 which preserve commutators and cubes. This situation is of interest for in characteristic 2 Jordan homomorphisms are the same thing as Lie homomorphisms, that is, mappings which preserve commutators. Lie mappings for matrices have been completely determined [5]. However little information is known for general simple rings.

Of particular interest is the type of argument used to establish the result for it uses the theory of Lie and Jordan ideals and substructures of simple rings developed by Herstein [3].

2. Main section. As is customary the commutator ab-ba will be denoted by $[a, b]$.

Initially φ will be a mapping from a simple ring R onto a simple ring $R' \neq 0$ of characteristic 2 which satisfies

(i) $$\varphi(x + y) = \varphi(x) + \varphi(y)$$

(ii) $$\varphi(z^2) = \varphi(z)^2$$

(iii) $$\varphi(z^3) = \varphi(z)^3$$

for all $x, y, z \in R$. Later we weaken (ii) to the assumption that $\varphi(xy - yx) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x)$.

Although we do not assume that the characteristic of R is 2, it can be easily proved. Clearly $\varphi(2x) = 0$ for all $x \in R$; but the kernel of φ is a Jordan ideal of R, and if the characteristic of R is not 2, the only non-zero Jordan ideal of R would be R itself [3]; thus $2x = 0$ for all $x \in R$ and R has characteristic 2.

Assume that φ is a mapping satisfying (i), (iii) and (ii)' $\varphi(xy+yx) = \varphi(x)\varphi(y) + \varphi(y)\varphi(x)$ for all $x, y \in R$, and that R is a field. From (ii)', R' is a field. On linearizing (iii) we find that

(1) $\varphi(xy(x + y)) = \varphi(x)\varphi(y)\varphi(x + y)$ for all $x, y \in R$. Let $W =$

Received July 30, 1959. The research of this paper was supported by an OOR contract with Cornell University.