ISOPERIMETRIC RATIOS OF REULEAUX POLYGONS

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1. Lebesgue [3] proved that, among all orbiforms of given breadth, the Reuleaux triangle has the least area and the circle, of course, the greatest area. Further, these are the only extremal figures. This paper is an elaboration of that result.

An orbiform is a convex body in the Euclidean plane which is such that the distance between parallel support lines (breadth) is constant. A Reuleaux polygon is an orbiform whose boundary consists of a finite number (greater than one) of circular arcs (sides). Reuleaux polygons necessarily have an odd number of sides. For details on these matters see [1]. If the sides are of equal length, the Reuleaux polygon is called regular. We shall prove that any two regular Reuleaux polygons having the same number of sides are similar. All Reuleaux triangles are regular. Our elaboration of Lebesgue's result is contained in the following three theorems.

THEORM 1. *The isoperimetric ratio (ratio of area to squared perimeter) of regular Reuleaux polygons strictly increases with the number of sides.*

THEOREM 2. *Among all Reuleaux polygons having the same number of sides, the regular Reuleaux polygons (and only these) attain the greatest isoperimetric ratio.*

THEOREM 3. *For any odd integer $n > 3$ and any $\varepsilon > 0$, there is an n-sided Reuleaux polygon whose isoperimetric ratio exceeds that of the Reuleaux triangle by an amount less than ε.*

As a matter of terminology, when reference is made in this paper to an n-sided Reuleaux polygon, it will always be understood that n is odd and that none of these sides is of zero length. As a matter of notation, $|PQ|$ means the length of segment PQ.

2. Being concerned only with isoperimetric ratios, we limit ourselve to Reuleaux polygons of unit breadth without loss of generality. The centre C from which a circular side of such a polygon is drawn must lie on the boundary of the polygon if the polygon is to be of constant breadth. Moreover, if C' and C'' are the end points of this side, then C is the junction of those sides centred at C' and C''. In proceeding in a positive direction along the boundary arc strictly between C' and C'', the support lines of the polygon turn through an angle of measure ϕ equal to the length of arc from C' to C'' which is the same as the

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