IRREDUCIBLE CONGRUENCE RELATIONS ON LATTICES

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1. Introduction. The structure of the lattice L is known to depend upon properties of the distributive lattice $\vartheta(L)$ of all congruence relations on L; for example,

(1.1) (Birkhoff [1]) L is a subdirect union of a finite number of simple lattices if and only if $\vartheta(L)$ is a finite Boolean algebra,

(1.2) (Dilworth [2]) L is a direct union of a finite number of simple lattices if and only if $\vartheta(L)$ is a finite Boolean algebra in which all the elements permute.

In the early development of structure theory for lattices, L was assumed to be modular, and the notion of projectivity was used to study congruence relations. For non-modular lattices a more general concept was needed; accordingly, Dilworth [2] devised the notion of weak projectivity and showed that complementation has a strong influence on structure. He proved:

(1.3) Every relatively complemented lattice satisfying the ascending chain condition is the direct union of a finite number of simple relatively complemented lattices;

(1.4) Every finite dimensional locally relatively complemented lattice is a subdirect union of a finite union of simple, locally relatively complemented lattices;

(1.5) A relatively complemented lattice which satisfies a chain condition is simple if and only if all prime quotients are projective.

More recently these results have been developed and generalized by Tanaka [7], Maeda [6], and Hashimoto [4].

It is interesting to observe for the lattices described in (1.3), (1.4), and (1.5), weak projectivity of prime quotients reduces to projectivity. The present paper studies the relationship between weak projectivity and projectivity of prime quotients. It is shown that if L satisfies the descending chain condition and if each join irreducible element of L covers some element, then the corresponding irreducible congruence relations generate $\vartheta(L)$ and provide simple criteria for the structure of L.

2. Definitions. This section contains definitions of the basic terms which are used; terminology generally conforms to that given in Birkhoff [1].

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