NORMAL SUBGROUPS OF SOME HOMEOMORPHISM GROUPS*

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1. Introduction. The normal subgroups of the group of all homeomorphisms of a space X have been enumerated by Fine and Schweigert [2] when X is a line, by Schreier and Ulam [3] when X is a circle, by Ulam and von Neumann [4] and Anderson [1] when X is a 2-sphere. In each of these cases there are either one or two proper normal subgroups. However, when X is an n-cell ($n > 1$), there are infinitely many. The object of this paper is to investigate the normal subgroups for a class of spaces X which includes the n-cell. Some of these normal subgroups, although not all, can be defined in terms of the family of fixed point sets of their elements, and we investigate this relationship at some length. A smallest normal subgroup is exhibited, and the corresponding quotient group is represented as a group of transformations of a related space.

2. Families of fixed point sets. Let X be a set, $\Pi(X)$ the group of all permutations of X (one-to-one mappings of X onto itself), and G a subgroup of $\Pi(X)$. Suppose that $\mathscr{F}$ is a non-empty family of subsets of X satisfying the following conditions:

(i) If $F_1, F_2 \in \mathscr{F}$, then there exists an $F_3 \in \mathscr{F}$ such that $F_3 \subset F_1 \cap F_2$,

(ii) If $F_1 \in \mathscr{F}$ and $g \in G$, then there exists an $F_2 \in \mathscr{F}$ such that $F_2 \subset g(F_1)$.

We shall call $\mathscr{F}$ ecliptic relative to G. For example, if $\mathscr{F}$ consists of the complements of all finite subsets of X, then $\mathscr{F}$ is ecliptic relative to $\Pi(X)$. If X has a topology, we denote the group of homeomorphisms of X by $H(X)$. Let X be a closed unit ball B_n in euclidean n-space and $\mathscr{F}_0$ consist of the complements in B_n of those balls which are concentric with B_n and have radius less than one. Then $\mathscr{F}_0$ is ecliptic relative to $H(B_n)$. In this connection, we note that for $h \in H(B_n)$, $h(S_{n-1}) = S_{n-1}$, where S_{n-1} is the boundary of B_n.

Let X again be an arbitrary set and G a subgroup of $\Pi(X)$. We introduce a partial ordering among the families of subsets of X as follows: $\mathscr{F} \leqq \mathscr{F}'$ provided that, for every $F \in \mathscr{F}$, there exists an $F' \in \mathscr{F}'$ such that $F' \subset F$. Evidently $\mathscr{F} \subset \mathscr{F}'$ implies $\mathscr{F} \leqq \mathscr{F}'$, where $\mathscr{F} \subset \mathscr{F}'$ means set inclusion, but the converse is false. We define equivalence of $\mathscr{F}$ and $\mathscr{F}'$ to mean $\mathscr{F} \leqq \mathscr{F}'$ and $\mathscr{F}' \leqq \mathscr{F}$, and we write $\mathscr{F} \cong \mathscr{F}'$.

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