ON THE REPRESENTATION OF OPERATORS BY CONVOLUTION INTEGRALS

J. D. WESTON

1. Introduction. Let $\mathfrak{X}$ be the complex vector space consisting of all complex-valued functions of a non-negative real variable. For each positive number u, let the *shift operator* I_u be the mapping of $\mathfrak{X}$ into itself defined by the formula

$$I_u x(t) = \begin{cases} 0 & (0 \leqq t < u) \\ x(t-u) & (t \geqq u) \end{cases}$$

Evidently, $I_{u+v} = I_u I_v$, for any positive numbers u and v.

A linear operator A which maps a subspace $\mathfrak{D}$ of $\mathfrak{X}$ into itself will here be called a *V-operator* (after Volterra) if

(1.1) for each x in $\mathfrak{D}$, the conjugate function x^* belongs to $\mathfrak{D}$,

(1.2) both $\mathfrak{D}$ and $\mathfrak{X}\backslash\mathfrak{D}$ are invariant under the shift operators,

(1.3) every shift operator commutes with A.

Many operators that occur in mathematical physics are of this type. If $\mathfrak{D}$ is any subspace of $\mathfrak{X}$ having the properties (1.1) and (1.2), the restriction to $\mathfrak{D}$ of each shift operator is an example of a V-operator. All 'perfect operators' (of which a definition may be found in [5][1]) are V-operators, on the space of perfect functions.

In this paper we obtain a representation theorem for V-operators which are continuous in a certain sense. This result leads to characterizations of two related classes of perfect operators, one of which has been considered from a different point of view in [5]. The main representation theorem (Theorem 4) is similar to a result obtained by R. E. Edwards [2] for V-operators which are continuous in another sense; and it closely resembles a theorem given recently by König and Meixner ([3], Satz 3).

2. Elementary properties of V-operators. An important property of V-operators is given by

THEOREM 1. *Let A be a V-operator, and let x_1 and x_2 be two of its operands such that, for some positive number t_0, $x_1(t) = x_2(t)$ whenever $0 \leqq t \leqq t_0$. Then $Ax_1(t) = Ax_2(t)$ whenever $0 \leqq t \leqq t_0$.*

Proof. Let $x = x_1 - x_2$. Then, since $x(t) = 0$ if $0 \leqq t \leqq t_0$, there is

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[1] And in § 4 below.