A CLASS OF LINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS

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I. Introduction. The purpose of this paper is to study the following integral equation:

$$\varphi(x) = \int_x^{x+1} K(y)\varphi(y)dy \tag{1}$$

or the differential-difference equation

$$\varphi'(x) = K(x+1)\varphi(x+1) - K(x)\varphi(x) \tag{1'}$$

with the boundary condition

$$\lim_{x\to\infty} \varphi(x) = 1 \ . \tag{2}$$

Equations of the type (1), (1′) have been investigated in great generality by many authors. In particular, the interested reader is referred to Yates [6], and Cooke [2], for recent developments, and a bibliography of significant earlier work. The equations of the form (1) which we shall consider are related to the class of linear differential-difference equations with asymptotically constant coefficients, a class treated thoroughly by Wright [5], and Bellman [1].

The novelty of the results below arises from the boundary condition (2) which appears not to have been studied before, and which gives reresults of an essentially different character from those of the works cited above. The system (1), (2) is of interest in some problems connected with the theory of neutron slowing down (Placzek [3]).

A further departure from previous work is the fact that no use is made of complex variable methods or the asymptotic characteristic equation of the kernel $K(y)$.

Aside from some fairly obvious theorems concerning uniqueness, boundedness and positivity, our main results are the following:

(a) necessary and sufficient conditions for the existence of a solution of (1), (2); this is achieved by constructing a minorant for the solution.

(b) proof of the existence of $\varphi(-\infty)$ under fairly general conditions.

(c) an application of Fubini's theorem to exhibit a rather surpris-

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