PARTITIONS OF MASS-DISTRIBUTIONS AND OF CONVEX BODIES BY HYPERPLANES

B. GRÜNBAUM

1. Introduction. The following results are well-known (Neumann [7]; Eggleston [3], [4, p. 125–126], [5, p. 118]; Newman [8]:

(A) For any mass-distribution in the plane, such that the total mass contained in every half-plane is finite and depends continuously on the position of the half-plane, there exists a point P such that each half-plane which contains P, contains at least 1/3 of the total mass.

(B) For any convex body K in the plane there exists a point P such that for each half-plane H containing P the area of $H \cap K$ is at least 4/9 of the area of K.

The main object of the present note is to generalize (A) and (B) to higher-dimensional Euclidean spaces.

In the following m shall denote a fixed (non-negative) finite measure on the ring of subsets of E^n generated by the closed half-spaces in E^n. (For the terminology and results on measures see, e.g., Halmos [6].)

For a real λ, $0 \leqq \lambda \leqq 1/2$, we define $\mathscr{C}(m, \lambda)$ as the subset of E^n consisting of those points $P \in E^n$ which satisfy the condition: For any closed half-space $H \subset E^n$, with $P \in H$, the relation $m(H) \geqq \lambda \cdot m(E^n)$ holds.

Obviously, $\mathscr{C}(m, \lambda)$ is a compact, convex (possibly empty) set.

Using the notation of $\mathscr{C}(m, \lambda)$, Theorem (A) may be extended as follows:

THEOREM 1. *$\mathscr{C}(m, 1/(n+1)) \neq \phi$ for any measure m in E^n.*

Let $V(S)$ denote the volume (n-dimensional Lebesgue measure) of the set $S \subset E^n$. For any convex body $K \subset E^n$, we denote by m_K the measure (defined for all Lebesgue measurable subsets S of E^n) obtained by taking $m_K(S) = V(S \cap K)$. We denote $\mathscr{C}(m_K, \lambda)$ by $\mathscr{C}(K, \lambda)$.

Theorem (B) may now be generalized as follows:

THEOREM 2. *If K is any convex body in E^n then*

$$\mathscr{C}\left(K, \left(\frac{n}{n+1}\right)^n\right) \neq \phi .$$

We shall prove Theorems 1 and 2 in the following two sections.

Received January 22, 1960. This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under contract No. AF49(638)-253. Reproduction in whole or in part is permitted for any purpose of the United States Government.