SINGULARITIES OF THREE-DIMENSIONAL HARMONIC FUNCTIONS

R. P. GILBERT

Introduction. Recently G. Szegö [9] and Z. Nehari [8] have obtained some interesting results connecting the singularities of axially symmetric harmonic functions with those of analytic functions. In this paper we shall show that a similar connection also exists between the singularities of a three-dimensional harmonic function and a function of two complex variables. We may do this by considering the Whittaker-Bergman operator [10] [1] $B_3(f, \mathscr{L}, X_0)$ which transforms functions of two complex variables $f(t, u)$, into harmonic functions of three variables.

$$H(X) = B_3(f, \mathscr{L}, X_0),\ B_3(f, \mathscr{L}, X_0) = \frac{1}{2\pi i}\int_{\mathscr{L}} f(t, u)\frac{du}{u}$$

$$t = \left[-(x - iy)\frac{u}{2} + z + (x + iy)\frac{u^{-1}}{2}\right],$$

$$|X - X_0| < \varepsilon, \qquad X \equiv (x, y, z),\ X_0 \equiv (x_0, y_0, z_0)\ ,$$

where $\mathscr{L}$ is a closed differentiable arc[1] in the u-plane, and $\varepsilon > 0$ is sufficiently small. We may see how this operator maps the functions $f(t, u)$ into harmonic functions by considering the homogeneous polynomials of degree n in x, y, z, which are defined by

$$t^n = \left\{-(x-iy)\frac{u}{2} + z + (x + iy)\frac{u^{-1}}{2}\right\}^n = \sum_{m=-n}^{+n} h_{n,m}(x, y, z)u^{-m}\ .$$

The $h_{n,m}(x, y, z)$ are linearly independent polynomials, which form a complete system [4]. Now, any harmonic function regular in a neighborhood of the origin $|X| < \varepsilon$, may be expanded into a series

$$H(X) = H(x, y, z) = \textstyle\sum_{n=0}^{\infty} \sum_{l=-n}^{+n} a_{n,l} h_{n,l}(x, y, z),$$

which converges inside the smallest sphere on whose surface there is a singularity of $H(X)$.

From the definition of the harmonic polynomials we see that

$$\frac{1}{2\pi i}\int_{\mathscr{L}} t^n u^m \frac{du}{u} = h_{n,m}(x, y, z)\ ,$$

where $\mathscr{L}$ is, say, the unit circle. In spherical coordinates this result may be recognized as one of Heine's [7] integral representations for the

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1 We shall usually consider $\mathscr{L}$ to be closed; however there is nothing preventing us from considering open arcs also.