INVOLUTIONS ON LOCALLY COMPACT RINGS

PAUL CIVIN

By a proper involution * on a ring R we mean a mapping $x \to x^*$ defined on R with the following properties:

(i) $(x+y)^* = x^* + y^*$,

(ii) $(xy)^* = y^*x^*$,

(iii) $(x^*)^* = x$ and

(iv) $xx^* = 0$ if and only if $x = 0$. If (iv) is not assumed, the mapping is simply termed an involution. If F is a field with an involution $\sharp$ and R is an algebra over F, we say that an involution on R is an algebra involution if in addition to (i)-(iv) above the following holds:

(v) $(\alpha x)^* = \alpha^\sharp x^*$ for all $x \in R$ and $\alpha \in F$.

We are concerned principally with involutions on two types of locally compact semi-simple rings, namely those which are compact or connected. The main result is that involutions on such rings are automatically continuous. As a byproduct we determine the form of any proper involution on a total matric ring R over a division ring. If in addition R is topological and the division ring admits only continuous involutions, then we note that R has only continuous involutions.

LEMMA *Let D be a division ring with center Z. Let R be a total matric ring over D. Any ring involution * on R induces an involution $\sharp$ on Z, and * is an algebra involution on R with respect to the involution $\sharp$ on Z.*

Direct calculation shows that the center of R consists of the totality of elements of the form αI where $\alpha \in Z$ and I is the identity of R. Suppose x is in the center of R and $y \in R$, then $x^*y = (y^*x)^* = (xy^*)^* = yx^*$, so x^* is in the center of R. Since $I^* = I$ is immediate, it follows that for any $\alpha \in Z$, there is a $\beta \in Z$ such that $(\alpha I)^* = \beta I$. Denote β by $\alpha^\sharp$. It is clear that $\sharp$ is an involution on Z. Moreover, if $\alpha \in Z$ and $x \in R$, $(\alpha x)^* = [(\alpha I)x]^* = x^*\alpha^\sharp I = \alpha^\sharp x$, so * is an algebra involution on R with respect to the involution $\sharp$ on Z.

THEOREM 2. *Let R be a total matric ring over D, where D is a division ring with center Z. Let * be a proper ring involution on R, and let $\sharp$ be the induced involution on Z. Then there exist a set of matrix units $\{g_{ij}\}$ in R such that $g_{ii}^* = g_{ii}$ and a set of non-zero elements γ_i of Z such that $\gamma_i^\sharp = \gamma_i$ such that the involution * has the following form: If $x = \sum \alpha_{ij}e_{ij}$, with $\alpha_{ij} \in D$, then $x^* = \sum \gamma_j^{-1}\alpha_{ij}\gamma_i e_{ji}$.*

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