ON EXPANSIVE HOMEOMORPHISMS

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1. Introduction. A homeomorphism ϕ of a compact metric space X onto X is said to be *expansive* provided there exists $d > 0$ such that if $x, y \in X$ with $x \neq y$, then there exists an integer n such that $\rho(x\phi^n, y\phi^n) > d$ (see [1] and [3]). The question arises as to the possibility of extending the results concerning expansive homeomorphisms to compact uniform spaces. The extension is possible, although trivial in light of the corollary to Theorem 1.

In §§ 3 and 4 the setting is a compact metric space X. Theorem 2 is stronger than Theorem 10.36 of [1] in that we do not require X to be self-dense. Also, the lemmas of which Theorem 2 is a consequence are perhaps of some interest in themselves. In § 4 we show that if X is self-dense, then for each $x \in X$ and each $\varepsilon > 0$ there exists $y \in U(\varepsilon, x)$ such that x and y are not doubly asymptotic.

2. A homeomorphism ϕ of a compact uniform space $(X, \mathscr{U})$ onto $(X, \mathscr{U})$ is said to be expansive provided there exists $U \in \mathscr{U}$ such that $U \neq \Delta$ (the diagonal) and if $x, y \in X$ with $x \neq y$, then there exists an integer n such that $(x\phi^n, y\phi^n) \notin \bar{U}$. For uniform spaces we use the notation of [2], but following Weil [4] we suppose $(X, \mathscr{U})$ is Hausdorff; i.e., $\cap\{U\colon U \in \mathscr{U}\} = \Delta$. We also suppose that each $U \in \mathscr{U}$ is symmetric.

THEOREM 1. *Let $(X, \mathscr{U})$ be a compact uniform space which is not metrizable and let ϕ be a homeomorphism of X onto X. If $U \in \mathscr{U}$, then there exist $x, y \in X$ with $x \neq y$ such that $(x\phi^n, y\phi^n) \in U$ for each integer n. (Compare with Theorem* 10.30 *of* [1].)

Proof. Select $V \in \mathscr{U}$ such that $V \circ V \circ V \subset U$ and $\bar{V} \subset U$ (see [2], p. 180). Since ϕ^n, for each integer n, is uniformly continuous, we may choose $U_1 \in \mathscr{U}$ with $U_1 \subset V$ such that $(p, q) \in U_1$ implies $(p\phi^k, q\phi^k) \in V$ for $k = \pm 1$. For $i > 1$, choose $U_i \in \mathscr{U}$ with $U_i \subset U_{i-1}$ such that $(p, q) \in U_i$ implies $(p\phi^k, q\phi^k) \in V$ for $k = \pm i$. Since $(X, \mathscr{U})$ is not metrizable, the countable set $\{U_i \mid i = 1, 2, \cdots\}$ is not a base for the uniformity $\mathscr{U}$([4], p. 16). Thus there exists $W \in \mathscr{U}$ with $W \subset U$ such that $i \geqq 1$ implies $U_i \cap \operatorname{comp} W \neq 0$. Choose, for each $i, (x_i, y_i) \in U_i \cap \operatorname{comp} W$. Since $X \times X$ is a compact Hausdorff space, there exists (x, y) such that each neighborhood of (x, y) contains (x_i, y_i) for an infinite number of positive integers i. Let n be an arbitrary positive integer, then there exists $m > n$ such that $(x_m, y_m) \in U_n(x) \times U_n(y)$. Hence $(x, x_m) \in U_n$ and $(y, y_m) \in U_n$;

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