MINIMAL SUPERADDITIVE EXTENSIONS OF SUPERADDITIVE FUNCTIONS

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Introduction. A real valued function f is said to be superadditive on an inverval $I = [0, a]$ if it satisfies the inequality $f(x + y) \geqq f(x) + f(y)$ whenever x, y and $x + y$ are in I. Such functions have been studied in detail by E. Hille and R. Phillips [1] and R. A. Rosenbaum [2]. In this paper we show that any superadditive function f on I has a minimal superadditive extension F to the non-negative real line E, and then proceed to show that F inherits much of its behavior from the behavior of f. We deal primarily with superadditive functions which are continuous and non-negative.

A simple example of a superadditive function on $[0, a]$ is furnished by a convex function f with $f(0) \leqq 0$. Also, if f is convex and $f(0) = 0$, then it is easy to verify that its minimal superadditive extension F is given by

$$F(x) = nf(a) + f(x - na)$$

for $na \leqq x < (n + 1)a$. In general, the minimal superadditive extension F is not easily computed. In the sequel we shall discuss two methods for obtaining F. For brevity we shall use the notation f^*F to mean "F is the minimal superadditive extension of f".

1. The decomposition method. Definition. Let $x \in E$. The numbers $x^1, \cdots, x^n$ are said to form an *a-partition* for x if $x^1 + \cdots + x^n = x$ and for each $i = 1, \cdots, n$ we have $0 \leqq x^i \leqq a$.

Theorem 1. *Let f be a superadditive function on $I = [0, a]$. Then the function F defined on E by the equation*

$$F(x) = \sup \Sigma f(u^i) ,$$

the supremum being taken over all a-partitions of x, is the minimal superadditive extension of f.

Proof. We will show that F is superadditive. The minimality of F will then follow from the fact that any superadditive extension $\hat{f}$ of f must satisfy $\hat{f}(x) \geqq \Sigma f(x^i)$ for all $x \in E$ and all a-partitions $x^1, \cdots, x^n$ of x. Let $x, y \in E$, $\varepsilon > 0$. Choose a-partitions $x^1, \cdots, x^m$ and $y^1, \cdots, y^n$ for

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