ON THE NILPOTENCY CLASS OF A GROUP OF EXPONENT FOUR

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Introduction. If G is a multiplicative group with elements $x, y, \cdots$, we define the commutator (x, y) by $(x, y) = x^{-1}y^{-1}xy$ and, inductively for length n, $(x_1, \cdots, x_{n-1}, x_n) = ((x_1, \cdots, x_{n-1}), x_n)$. We also use the notation $(x, \cdots, y; \cdots; z, \cdots, w)$ for the commutator $((x, \cdots, y), \cdots, (z, \cdots, w))$. For each positive integer n, let G_n be the subgroup of G generated by all commutators of length n.

A group, G, is of exponent 4 in case $x^4 = 1$ for every x in G but $y^2 \neq 1$ for some y in G. Let F be a free group of rank k, and let F^4 be the subgroup generated by fourth powers of elements of F. Let $B(k) = F/F^4$. Then $B(k)$ is clearly a group of exponent 4 on k generators. Moreover, every group of exponent 4 on k generators is a homomorphic image of $B(k)$.

I. N. Sanov has shown that $B(k)$ is finite. (See [2], pp. 324–325, or [3]). Unfortunately, his proof gives very little additional information about $B(k)$. The present paper is devoted to the study of relations between commutators in the group $B(k)$, a consequence of the relations obtained being that $B(k)_{3k} = 1$.

Preliminaries. Let G be a group of exponent 4, and let $a, b, \cdots$ be elements of G. Then

$$(1) \qquad (a, b)^2 \equiv (a, b, b, b)(a, b, b, a)(a, b, a, a) \bmod G_4$$

$$(2) \qquad (a, b, a)^2 = (a, b, a, a, a) = (a, b, a; a, b)$$

$$(3) \qquad (a, b, c) \equiv (b, c, a)(c, a, b) \bmod G_4$$

$$(4) \qquad (a, b; c, d) \equiv (a, c; b, d)(a, d; b, c) \bmod G_5$$

$$(5) \qquad (\mathbf{a}, b; \mathbf{c}, d; f) \equiv (\mathbf{a}, d; \mathbf{c}, f; b)(\mathbf{a}, f; \mathbf{c}, b; d) \bmod G_6$$

where the bold-face type in (5) has no significance other than to point out which entries are left fixed while the others are cyclicly permuted—whenever bold-face type appears in a computation an application of (5) is about to be made. The relations (1) and (2) can be shown to hold in $B(2)$; hence they certainly hold in any group, G, of exponent 4. Relation (3) is simply the Jacobi identity (which holds in any group) adapted to exponent 4. Relations (4) and (5) were proved in [4] for the case in which the entries are of order 2, but the proofs clearly go through without this restriction, since in proving the relations we are simply

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