ABSTRACT MARTINGALE CONVERGENCE THEOREMS

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Introduction. The study of probability theory in abstract spaces became possible with the introduction of integration theories in such spaces. Thus the idea of the expectation of a random variable which takes its value in a Banach space was studied by Frechet [6] with what amounted to the Bochner integral, and by Mourier [13] with the Pettis integral. Doss [2] studied the problem in a metric space. Kolmogorov [10] generalized the notion of characteristic function. Generalizations of the laws of large numbers and the ergodic theorem appear in Mourier [13] and Fortet-Mourier [5]. In this paper we generalize the concept of martingale and prove various convergence theorems.

Chapter I is devoted to listing various definitions and theorems which we shall have to refer to later. In Chapter II we introduce the idea of the conditional expectation of a Banach space valued random variable. We also prove the existence of the strong conditional expectation for strongly measurable random variables. This part of our work was also done by Moy [14] independently, and without the knowledge of the author. Chapter III is devoted to the definition and study of weak and strong $\mathfrak{X}$-martingales, with emphasis on the latter.

In Chapter IV we prove a series of convergence theorems for $\mathfrak{X}$-Martingales with the help of theorems of Doob [1]. The main theorem says that if $\{x_n, \mathscr{F}_n, n \geqq 1\}$ is an $\mathfrak{X}$-Martingale where $\mathfrak{X}$ is a reflexive Banach space, and if $\{\|x_n\|, n \geqq 1\}$ is a uniformly integrable class of functions, then there is a strongly measurable $\mathfrak{X}$-valued function x_∞ such that $\|x_n(\omega) - x_\infty(\omega)\| \to 0$ as $n \to \infty$ with probability 1 and $\{x_n, \mathscr{F}_n, 1 \leqq n \leqq \infty\}$ is an $\mathfrak{X}$-martingale. We close by discussing examples where $\mathfrak{X}$ is one of the standard Banach spaces, l^p, $L^p(I)$, and $C(I)$.

CHAPTER I.

PRELIMINARY DEFINITIONS

1. Measurability concepts. A. Let $(\Omega, P, \mathscr{M})$ be a probability space. Thus Ω is an abstract set of points ω, $\mathscr{M}$ is a Borel field of subsets of Ω, and P is a probability measure defined on $\mathscr{M}$. We recall that a Borel field of sets is a class of sets which is closed under countable unions and intersections, and complementation. A probability

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