# APPLICATIONS OF THE SUBORDINATION PRINCIPLE TO UNIVALENT FUNCTIONS

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**1. Introduction.** Let

$$f(z) = z + a_2z^2 + \cdots + a_nz^n + \cdots \tag{1.1}$$

be regular and univalent in $|z| < 1$ and map $|z| < 1$ onto a simply-connected domain $D$. Let

$$\phi(z) = b_1z + b_2z^2 + \cdots + b_nz^n + \cdots \tag{1.2}$$

also be regular in $|z| < 1$. $\phi(z)$ is said to be subordinate to $f(z)$ if for each $z$ of the unit circle $|z| < 1$ the corresponding point $w = \phi(z)$ lies in the domain $D$. In this case [2] there exists an analytic function $\omega(z)$ regular in $|z| < 1$ for which $\omega(0) = 0$, $|\omega(z)| \leqq |z| < 1$ and $\phi(z) \equiv f\{\omega(z)\}$.

It is the purpose of this paper to establish the following basic Theorems A and B which concern analytic functions $F(z, t)$ and $\omega(z, t)$, depending upon a real parameter $t$, and then to use them to obtain results in the theory of univalent functions. Some of the results are well known and others are new, but the method of attack seems to be novel, simple and of sufficient generality to be of interest in itself. The functions $F(z, t)$ and $\omega(z, t)$ will be related to the univalent function $f(z)$ of (1.1) by means of the subordination concept.

An interesting biproduct of Theorem B is the following statement. A sufficient condition that $f(z)$, regular and univalent in $|z| < 1$, be convex in $|z| < 1$ is that the de la Vallée Poussin means $V_n(z)$ of (1.1) be subordinate to $f(z)$ in $|z| < 1$ for $n = 1, 2, \cdots$. Recently [3] G. Pólya and I. J. Schoenberg showed that this condition for convexity is also necessary.

THEOREM A. *Let*

$$\omega(z, t) = \sum_1^\infty b_n(t)z^n \tag{1.3}$$

*be regular in* $|z| < 1$ *for* $0 \leqq t \leqq 1$. *Let*

$$|\omega(z, t)| < 1 \text{ for } |z| < 1, 0 \leqq t \leqq 1, \omega(z, 0) \equiv z .$$

*Let* $\rho$ *be a positive real number for which*

$$\omega(z) = \lim_{t \to 0+}\left\{\frac{\omega(z, t) - z}{zt^\rho}\right\} \tag{1.4}$$

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