

AN INEQUALITY FOR LOGARITHMIC CAPACITIES

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1. Introduction. In his work on capacities, G. Choquet proved that for many capacities the inequality of strong subadditivity holds [1]. It is the purpose of this note to show that a similar inequality holds for logarithmic capacities. More precisely we shall prove the

THEOREM. *Let A and B be compact sets in the complex z -plane E . By $C(S)$ we denote the logarithmic capacity [2] of a given compact set S , $S \subset E$, where we agree to put $C(S) = 0$ whenever $S = \phi$. Then*

$$C(A \cup B) \cdot C(A \cap B) \leq C(A) \cdot C(B).$$

2. Proof of the theorem. Let $S, S \subset E$, be a compact set whose boundary consists of a finite number of analytic arcs. By S^* we denote that component of $E - S$ which is unbounded. Then Green's function of S^* is defined by the properties: it is harmonic in S^* , vanishes at the finite boundary points of S^* and has a logarithmic singularity at infinity. We will denote this function by $g_S(z, \infty)$.

First we shall deal with the case when the respective boundaries of A, B and $A \cap B$ consist of a finite number of non-degenerate analytic arcs. We remark that the difference $g_{A \cap B}(z, \infty) - g_A(z, \infty)$ is harmonic in A^* , $A^* \subset (A \cap B)^*$, and at infinity. It is furthermore non-negative on the boundary of A^* and hence non-negative in A^* by the maximum principle. Similarly $g_{A \cup B}(z, \infty) \geq g_B(z, \infty)$ holds in B^* , $B^* \subset (A \cap B)^*$.

The function

$$h(z) = g_{A \cup B}(z, \infty) + g_{A \cap B}(z, \infty) - g_A(z, \infty) - g_B(z, \infty)$$

is harmonic in $(A \cup B)^*$ and at infinity. From $(A \cup B)^* = A^* \cap B^*$ it follows that the boundary points of $(A \cup B)^*$ belong either to the boundary of A^* or to the boundary of B^* . Therefore $g_{A \cup B}(z, \infty)$ and either $g_A(z, \infty)$ or $g_B(z, \infty)$ vanish at these boundary points. With the aid of the remark made above we get the result that $h(z)$ is non-negative in $(A \cup B)^*$.

Therefore

$$g_A(z, \infty) + g_B(z, \infty) \leq g_{A \cup B}(z, \infty) + g_{A \cap B}(z, \infty)$$

holds in $(A \cup B)^*$. From this general inequality and using the fact that

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