THE STONE-WEIERSTRASS PROPERTY IN BANACH ALGEBRAS

YITZHAK KATZNELSON AND WALTER RUDIN*

Introduction. Let A be a semi-simple commutative Banach algebra with maximal ideal space Δ. Regarding the elements of A as functions on Δ, we call a subalgebra B of A *self-adjoint* if corresponding to every $f \in B$ the function $\bar{f}$ defined on Δ by $\bar{f}(x) = \overline{f(x)}$ is also in B; we call B *separating* if to every pair of distinct points $x_0, x_1 \in \Delta$ there is an $f \in B$ such that $f(x_0) = 0$, $f(x_1) = 1$.

If every separating self-adjoint subalgebra of A is dense in A, we say that *A has the Stone-Weierstrass property.*

The Stone-Weierstrass property is related, to some extent at least, to the ideal structure of A. For instance, it is obvious that if A has a unit and a closed primary ideal I which is not maximal, then the algebra generated by I and the constants is not dense in A. More generally, suppose A is self-adjoint, I is a closed self-adjoint ideal in A which is not the intersection of the regular maximal ideals containing it, and A/I is the direct sum of its radical and a subalgebra B_0. If h is the canonical homomorphism of A onto A/I, then $I + h^{-1}(B_0)$ is a separating self-adjoint subalgebra of A which is not dense in A, so that A does not have the $S - W$ property.

Also, it was pointed out by Herz that the Schwartz counterexample [9] to spectral synthesis in $L^1(R^3)$ yields immediately an example of a closed, separating, self-adjoint, *proper* subalgebra of $L^1(R^3)$. After Malliavin's solution of the spectral synthesis problem for $L^1(\Gamma)$, where Γ is any locally compact abelian group, it was natural to investigate the $S - W$ property for these algebras.

In Part I (whose contents were announced in [5]) this is done for $\Gamma = Z$, the additive group of the integers. The general case is settled in Part II; the solution shows that the relation between the $S - W$ property and the ideal structure is, after all, not a very close one. Part III deals with the relation between the self-adjointness of A and the total disconnectedness of Δ.

For convenience of notation, we shall phrase our results on group algebras in $A(G)$ rather than in $L^1(\Gamma)$. Here G and Γ are dual groups of each other, and $A(G)$ is the algebra of all Fourier transforms of functions in $L^1(\Gamma)$. The circle group (the dual of Z) will be denoted by T, so that $A(T)$ is the algebra of all absolutely convergent Fourier series.

Since every locally compact abelian group is locally isomorphic to a

Received April 4, 1960. * Research Fellow of the Alfred P. Sloan Foundation.