RELATIVE HERMITIAN MATRICES

MAGNUS R. HESTENES

1. Introduction. The purpose of the present paper is to develop a spectral theory for an arbitrary $m \times n$ dimensional matrix A, which is analogous to that given in the hermitian case and which reduces to the usual spectral theory when A is hermitian. The theory is centered around the triple product AB^*C of matrices of the same dimension. Here B^* is the transpose of B in the field of real numbers and the conjugate transpose of B in the field of complex numbers. The matrix T will be said to be elementary in case $T = TT^*T$. Elementary matrices play the role of units and in case of vectors are unit vectors. Given an elementary matrix T and a matrix A of the same dimension the matrix TA^*T can be considered to be the conjugate transpose of A relative to T. If $A = TA^*T$, then A is hermitian relative to T. The polar decomposition theorem for matrices implies that to each matrix A there is a unique elementary matrix R such that A is hermitian relative to R, AR^* is non-negative hermitian in the usual sense, and R has the same null space as A. Every elementary matrix T relative to which A is hermitian is of the form $T = T_0 + R_1 - R_2$, where $R_1 + R_2 = R$ and T_0, R_1, R_2 are mutually $*$-orthogonal. Two matrices A and B are $*$-orthogonal in case $AB^* = 0$ and $A^*B = 0$. A matrix B will be called a section of A if B and $A - B$ are $*$-orthogonal.

If A is hermitian relative to an elementary matrix T, it is shown below that A and T can be written as sums of sections

$$A = A_1 + \cdots + A_q\,, \qquad T = T_1 + \cdots + T_q$$

such that $A_i = \lambda_i T_i$, where λ_i is a real number. Moreover these sections can be chosen so that $\lambda_i \neq \lambda_j$, $(i \neq j)$. If in this event the decomposition is unique. If $AT^* \geqq 0$, then $\lambda_i \geqq 0$. If in addition A and T have the same null space then $\lambda_i > 0$. In the event T is the identity, this result gives the usual spectral representation of hermitian matrices.

A matrix A will be said to be normal relative to an elementary matrix T in case $A = AT^*T = TT^*A$, $AA^*T = TA^*A$. In this event the spectral decomposition theorem described above holds, the coefficients λ_i being complex instead of real.

In the development of the theory the concept of $*$-commutativity of two matrices plays a significant role. The matrices A and B will be said to $*$-commute (see § 4 below) in case $AB^* = BA^*$ and $A^*B = B^*A$. If A and B $*$-commute, there is an elementary matrix T relative to

Received April 26, 1960. The preparation of this paper was sponsored by the Office of Naval Research and the Office of Ordnance Research, U. S. Army. Reproduction in whole or in part is permitted for any purpose of the United States Government.