ON THE ACTION OF A LOCALLY COMPACT GROUP ON E_n

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It is known [2, p. 208] that if a locally compact group acts effectively and differentiably on E_n then it is a Lie group. The object of this note is to show that if the differentiability requirements are replaced by some weaker restrictions, given later on, the theorem is still true. Let G be a locally compact group acting on E_n and let the coordinate functions of the action be given by $f_i(g, x_1, \cdots, x_n)$, $1 \leqq i \leqq n$. For economy we introduce the following notation

$$Q_{ij}(g, t, x) = \frac{f_i(g, x_1, \cdots, x_j + t, \cdots, x_n) - f_i(g, x_1, \cdots, x_j, \cdots, x_n)}{t} .$$

We denote by $\sigma(Q_{ij}(e, 0, x))$ the oscillation of $Q_{ij}(g, t, x)$ at the point $(e, 0, x)$.

Before proceeding there is one simple remark to be made on matrices. If $A = (a_{ij})$ is an $n \times n$ matrix such that $| a_{ij} - \delta_{ij} | < (1/n)$ then A is non-singular. If A were singular there would be a vector x such that $\sum_i x_i^2 = 1$ and $Ax = 0$. From the Schwarz inequality it follows that $x_i^2 = (\sum_j (a_{ij} - \delta_{ij}) x_j)^2 < (1/n)$ and consequently $1 = \sum x_i^2 < 1$ which is impossible. If $| a_{ij} - \delta_{ij} | \leqq (\alpha/n)$, where $0 < \alpha < 1$, then the determinant of A is bounded away from zero since the determinant is a continuous function and the set $\{a_{ij}: | a_{ij} - \delta_{ij} | \leqq (\alpha/n)\}$ is compact in E_{n^2}.

THEOREM 1. *If T is a pointwise periodic homeomorphism of E_n then T is periodic.*

Proof. [2, p. 224.]

THEOREM 2. *If G is a compact, zero dimensional, monothetic group acting effectively on E_n and satisfying*

$$(*) \qquad \sigma(Q_{ij}(e, 0, x)) < \frac{\varepsilon}{n}, \quad 0 < \varepsilon < 1, \quad \textit{for each } x \textit{ in } E_n;$$

then G is a finite cyclic group.

Proof. Since G is monothetic, let a be an element whose powers are dense in G. It is enough to show that there is a power of a which leaves E_n pointwise fixed since the action of G is effective.

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