ON A CONJECTURE OF H. HADWIGER

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1. For any convex body (i.e., compact convex set with interior points) K in the Euclidean plane E^2 let $i(K)$ denote the greatest integer with the following property:

There exist translates K_n, $1 \leqq n \leqq i(K)$, of K such that

$$(1) \qquad \begin{array}{ll} K \cap K_n \neq \phi & \text{for all } n; \\ \operatorname{Int} K_n \cap \operatorname{Int} K_m = \phi & \text{for } n \neq m. \end{array}$$

It is well known (see e.g., Hadwiger [3]) that $7 \leqq i(K) \leqq 9$ for any $K \subset E^2$,[1] and that the bounds are attained (e.g., $i(K) = 7$ if K is a circle, $i(K) = 9$ if K is a parallelogram). Hadwiger conjectured,[2] moreover, that if K is not a parallelogram, then $i(K) = 7$.

We shall establish Hadwiger's conjecture in the following theorem:

If K is not a parallelogram, then $i(K) = 7$. Moreover, if 7 translates o K satisfy conditions (1) *then one of them coincides with K.*

In the proof we shall use some results on centrally symmetric convex sets; they are collected in § 2. The proof of the theorem follows in § 3. In § 4 we make some remarks on related problems in higher-dimensional spaces. § 5 contains some results on the related problem on the number of translates of a convex set needed to "enclose" the set.

2. Let K be any centrally symmetric plane convex body with the origin 0 as center. Then a Minkowski geometry, with norm $\| \ \|$, is defined in the plane, for which K is the unit cell.

We note the following propositions:

(i) *For any point x with $\| x \| = 1$ there exist points y, z satisfying $\| y \| = \| z \| = \| x - y \| = \| y - z \| = \| x + z \| = 1$.* (In other words, any $x \in \text{Front } K$ is a vertex of at least one affine-regular hexagon whose vertices belong to Front K).

(ii) *Let x, y, z be different points belonging to Front K, such that the origin 0 does not belong to that open half-plane determined by x and y which contains z. Then $\| x - y \| \geqq \| x - z \|$, with equality taking place only in case y, z, and $(y - x)/\| y - x \|$ belong to a straight-line segment contained in Front K.*

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[1] Related results, pertaining to more general sets, are given in [4].

[2] Oral communication from Dr. H. Debrunner.