# WEAK COMPACTNESS AND SEPARATE CONTINUITY

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1. For a locally compact space $X$ let $C(X)$ denote the Banach space of all bounded continuous complex valued functions on $X$, $C_0(X)$ the subspace of functions vanishing at infinity, so that the adjoint $C_0(X)^*$ consists of all finite complex regular Borel measures on $X$. In a natural fashion, we may view $C(X)$ as a subspace of $C_0(X)^{**}$.

When $X$ is compact Grothendieck [6; Th. 5] has shown that a bounded set $K \subset C(X)$ is compact in the weak topology if (and of course only if) $K$ is compact in the topology of pointwise convergence on $X$, and then both topologies, being comparable, coincide on $K$. In some recent work the writer was led to a simple corollary of Grothendieck's result which yields the significance, when $X$ is only locally compact, of compactness in $C(X)$ under pointwise convergence:

1.1. *Let $K$ be a bounded subset of $C(X)$, $X$ locally compact. Then $K$ is compact in the topology of pointwise convergence on $X$ (if and) only if $K$ is compact in the weak* topology of $C_0(X)^{**}$* [4, 5.1].

Again both topologies coincide on $K$. A direct corollary of 1.1 is

1.2. *Let $X$ and $Y$ be locally compact spaces, and $f$ a bounded complex function on $X \times Y$ which is separately continuous, i.e., for which all the maps*

$$x \to f(x, y) \text{ and } y \to f(x, y)$$

*are continuous. Then for $\mu \in C_0(X)^*$,*

$$y \to \int f(x, y)\mu(dx)$$

*is continuous* [4, 5.2].

The continuity obtained in 1.2 allows one to form the iterated integral

$$\iint f(x, y)\mu(dx)\nu(dy), \qquad \mu \in C_0(X)^*, \ \nu \in C_0(Y)^*, \tag{1.21}$$

and thus one can extend the notion of convolution of a pair of finite measures to a locally compact semigroup $S$ in which the operation is only separately continuous. Moreover 1.2 shows (1.21) is identical with

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