AN EMBEDDING OF RIEMANN SURFACES OF GENUS ONE

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The C^k embedding of a Riemann surface S will mean here the construction of a C^k surface S' in 3-space which is conformally equivalent to S, if angles on the surface S' are measured in the natural way.[1] The result to be obtained is:

THEOREM. *Any compact Riemann surface of genus one can be C^∞ embedded in 3-space.*

As is well known, any Riemann surface of genus one is conformally equivalent to a parallelogram in the plane with opposite sides identified. The method used here utilizes surfaces which are approximately isometric to the canonical surfaces determined by parallelograms. The parallelogram for a given conformal class may be picked in a standard way. We may take the vertices at the points $0, 2\pi, \omega, \omega + 2\pi$ in the complex plane. Then the parallelogram is determined by a single complex number ω. For any surface S conformally equivalent to this parallelogram with opposite sides identified, ω will be called a modulus of S, and the parallelogram a fundamental parallelogram of S. ω is not completely determined. A complete set of inequivalent canonical surfaces corresponds to the values of $\omega = \theta + i\lambda$ in the region

$$(1) \qquad -\pi < \theta \leqq \pi, \quad \theta^2 + \lambda^2 > 4\pi^2$$

or

$$0 \leqq \theta \leqq \pi, \quad \theta^2 + \lambda^2 = 4\pi^2 \,.$$

For each value of ω in this region a surface is needed.

A torus has a pure imaginary modulus which is easily computed. More generally, any surface with a plane of symmetry has pure imaginary modulus. Thus there are many ways in which one can construct a family of surfaces whose moduli fill the line $\theta = 0, \lambda \geqq 2\pi$.

For finding surfaces with $\theta \neq 0$, we may note first that under a reflection of space a surface with modulus $\theta + i\lambda$ is transformed into one with modulus $-\theta + i\lambda$. This means that if surfaces whose moduli represent all points of the region

Received February 23, 1960. This research was performed while the first author was supported by the United States Air Force under Contract No. AF 49(638)-42, monitored by the Air Force Office of Scientific Research of the Air Research and Development Command.

[1] In our considerations compact surfaces of 3-space will be considered oriented by the outward pointing normal.