ASYMPTOTICS II: LAPLACE'S METHOD FOR MULTIPLE INTEGRALS

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Laplace's method is a well known and important tool for studying the rate of growth of an integral of the form

$$I(h) = \int_a^b e^{-hf} g dx$$

as $h \to \infty$, where f has a single minimum in $[a, b]$. It's extension to multiple integrals has been studied by L. C. Hsu in a series of papers starting in 1948, and by P. G. Rooney (see bibliography). These authors establish what amount to a first term of an asymptotic expansion. All but one (see [7]) of these results are under fairly heavy smoothness conditions.

In this paper we examine multiple integrals of the form

$$I(h) = \int_R e^{-hf} g dx$$

where f and g are measurable functions defined on a set R in E_p. Without making any smoothness assumptions on f and g, and using only the existence of $I(h)$ and, of course, asymptotic expansions of f and g near the minimum point of f we obtain an asymptotic expansion of I. The special features of our procedure are the lack of smoothness assumptions and the fact that we get a complete expansion.

Without loss of generality we may assume that the essential infimum of f occurs at the origin, and that this minimal value is zero. We introduce polar coordinates: $x = (\rho, \Omega)$ where

$$\rho = |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_p^2}\ ,$$

and where $\Omega = x/|x|$ is a point on the surface, S_{p-1}, of the unit sphere.

Our hypothesis are the following:

(1) The origin is an interior point of R.

(2) For each $\rho_0 > 0$ there is an $A > 0$ such that $f(\rho, \Omega) \geqq A$ if $\rho \geqq \rho_0$. (This says that f can be close to zero only at the origin.)

(3) There is an $n \geqq 0$ and $n + 1$ continuous functions $f_k(\Omega)$, $k = 0, 1, 2, \cdots, n$, defined on S_{p-1} with $f_0 > 0$ for which

$$f(\rho, \Omega) = \rho^\nu \sum_{k=0}^{n} f_k(\Omega)\rho^k + o(\rho^{n+\nu}) \text{ as } \rho \to 0$$

Received April 29, 1960. The work on this paper was performed under sponsorship of the Office of Naval Research, Contract Nonr 710 (16), at the University of Minnesota.