MANIFOLDS WITH POSITIVE CURVATURE

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0. Introduction and a conjecture. In 1936 J. L. Synge [10] proved that an even dimensional orientable compact manifold M_n with positive sectional curvature is simply connected. His proof was an application of a formula for the second variation of arc length derived by him in an earlier article.[1] In the present paper we continue the study of positively curved manifolds again using the ideas of Synge and applying them to an only slightly different situation, namely to the "position" of certain submanifolds of M.

Theorem 1 states that two compact totally geodesic (see § 2 for definitions) submanifolds V_r and W_s of M_n must necessarily intersect if their dimension sum is at least that of M, i.e. if $r + s \geqq n$. As remarked above the proof is a straightforward continuation of Synge's method. Unfortunately totally geodesic submanifolds are not a too common occurrence.

If M_n is a Kähler manifold[2] the situation is much more satisfactory. There, instead of requiring V and W to be totally geodesic, we need only ask that they be complex analytic submanifolds (Theorem 2).

Examples of compact Riemannian manifolds of positive sectional curvature are the spheres, the real, complex and quaternionic projective spaces and the Cayley plane. Rauch [8] has shown that if the sectional curvatures do not differ too much from that of the sphere and if the space is simply connected, then it is itself topologically a sphere (see also the recent improvements by W. Klingenberg, Über kompakte Riemannsche Mannigfaltigkeiten, Math. Ann., **137** (1959), pp. 351–61). Berger [2] has shown that if M_n is an even dimensional, simply connected manifold and if the sectional curvature K satisfies $1/4 \leqq K \leqq 1$, then the manifold is one of the spaces listed above.

In the list the only Kähler manifolds are the complex projective n-spaces $P_n(\boldsymbol{C})$ with the usual Fubini metric. If (e_1, e_2) is a pair of orthonormal tangent vectors to $P_n(\boldsymbol{C})$, then the sectional curvature $K(e_1, e_1)$ satisfies $1/4 \leqq K(e_1, e_2) \leqq 1$ with $K = 1$ if and only if the plane $e_1 \wedge e_2$ is a "complex direction." It may very well be that

CONJECTURE. *The positively curved Kähler manifolds of complex dimension n are analytically homeomorphic to* $P_n(\boldsymbol{C})$. The Gauss Bonnet

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[1] For completeness we include in § 1 a derivation of the second variation formula.

[2] Since the Ricci curvature of a positively curved manifold is positive, the Kähler manifold is a "Hodge manifold" and Kodaira's theorem [6] states that the manifold is algebraic.