RANDOM CROSSINGS OF CUMULATIVE DISTRIBUTION FUNCTIONS

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1. Introduction. Let $X_1, \cdots, X_n$ be n independent and identically distributed random variables, each with continuous c.d.f. (cumulative distribution function), $F(x)$. Let $F_n(x)$ be the empirical c.d.f. of the n random variables and let $N_1(n)$ be the number of times F_n equals F. There is no loss of generality in supposing that the X_i's are distributed uniformly over the interval $(0, 1)$, and to be specific, $N_1(n)$ is defined by

$$N_1(n) = \text{number of indices, } i, \text{ for which } F_n(i/n) = i/n, \quad i = 1, \cdots, n.$$

Similary, let $X_1, \cdots, X_n, \cdots, Y_1, \cdots, Y_n$ be $2n$ independent random variables, each with the same continuous c.d.f., $F(x)$, and let F_n, G_n denote the empirical c.d.f.'s of the X_i's and Y_i's respectively. Let $N_2(n)$ be the number of times F_n equals G_n. That is.

$$\begin{aligned} N_2(n) = \text{number of indices } i \text{ for which } F_n(X_i) = G_n(X_i),& \\ \text{plus}& \\ \text{number of indices } i \text{ for which } F_n(Y_i) = G_n(Y_i),& \quad i = 1, \cdots, n. \end{aligned}$$

The purpose of this paper is to show that

$$\lim_{n\to\infty} P\Big(\frac{N_1(n)}{\sqrt{2n}} < t\Big) = \lim_{n\to\infty} P\Big(\frac{N_2(n)}{\sqrt{4n}} < t\Big) = 1 - e^{-t^2}.$$

The methods for obtaining these results are practically the same for N_1 and N_2, so the first case is treated with somewhat greater detail. In both cases, the random variables are related to other random variables on appropriate stochastic processes with independent increments, to obtain generating functions for the moments of N_i. The Karamata Tauberian theorem is then applied to describe the asymptotic behavior of these moments.

2. Some preliminaries on the Poisson process. Let $Y(t)$ be the Poisson process with stationary independent increments, $t \geqq 0$, $Y(0) = 0$, $E[Y(1)] = 1$. Consider γt, the straight line coming out of the origin with slope $\gamma > 1$. The random function $Y(t)$ can equal γt at times $1/\gamma$, $2/\gamma$, etc. The event that $Y(t) = \gamma t$ is a recurrent event in the sense of Feller [4]. Because γ is greater than 1, this recurrent event is an *uncertain* one. It was shown by Baxter and Donsker [1] that

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