MULTIPLICATION OPERATORS

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1. Introduction. The prototype for partially ordered linear spaces is $C[X]$, the space of all real valued continuous functions on a topological space X, with the natural ordering defined by: $f \geqq 0$ if and only if $f(x) \geqq 0$ for all $x \in X$. If V is a real linear space with a partial order defined by a suitable positive cone P, then V has a canonical embedding in a function space $C[X]$.

The containing space $C[X]$ has a more elaborate structure than did the original space V; in particular, $C[X]$ is an algebra. If we take any aspect of $C[X]$, we may ask how it appears when transferred back to V. This paper deals with one aspect of this.

Among the linear operators on $C[X]$, an interesting class that arises in many contexts is the class of multiplication operators. These are defined by:

$$T(f) = g \quad \text{where} \quad g(x) = \phi(x)f(x) \qquad x \in X ,$$

and where ϕ is a specific member of $C[X]$.

The central result in this paper is a simple characterization, in terms of order, of the linear operators on V which become multiplication operators when V is represented in a function space $C[X]$. This in turn yields a new and more transparent proof of the Stone-Krein theorem on ordered algebras.

2. A simpler case. Let V be a real linear space. We assume that there is a convex cone P with vertex at 0 which defines an order relation $\leqq$ in V by $x \leqq y$ if and only if $y - x \in P$. On P, we impose three conditions:

(1) $\quad P \cap -P = \{0\}$

(2) $\quad P$ is generating

(3) $\quad P$ is linearly closed in V.

The second condition implies that every element $x \in V$ is the difference of positive elements; the third condition requires that every line meet P in a (possibly unbounded) closed interval. Note that we do not impose any further lattice properties on V, nor do we assume that there is an order unit. If V' denotes the dual space of V, consisting of all linear functionals on V, then V' has a natural partial ordering derived from that of V. A functional L is said to be positive if $L(x) \geqq 0$ for

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