SYMMETRY IN GROUP ALGEBRAS OF DISCRETE GROUPS

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1. Introduction. The Banach algebras $\mathfrak{A}$ considered here are over the field of complex numbers, and have isometric involutions *. The involution is said to be *hermitian* if for any $x = x^* \in \mathfrak{A}$, the spectrum $Sp_{\mathfrak{A}}(x)$ of x contains only real numbers. The algebra $\mathfrak{A}$ is said to be *symmetric* if for any $y \in \mathfrak{A}$, $Sp_{\mathfrak{A}}(y^*y)$ contains only nonnegative real numbers.

A familiar example of a Banach algebra with an involution is the group algebra over the complex numbers of a locally compact group G. This is obtained by taking the Banach space $L^1(G)$ of all complex valued absolutely integrable functions with respect to the left invariant Haar measure dx on G. Multiplication is defined as convolution, and the involution by the formula $x^*(g) = \overline{x(g^{-1})}\rho(g)$, where $x \in L^1(G)$ and $\rho(\cdot)$ is the modular function relating the given measure to the right invariant measure by $dx^{-1} = \rho(x)dx$. This involution will be called the *natural involution* of the group algebra, and is the only involution on the group algebra we will consider.

It is known that when the group G is either compact or commutative, then its group algebra with respect to the natural involution is symmetric. On the other hand, in 1948 Neumark [6] showed that the natural involution in the group algebra of the homogeneous Lorentz group is not hermitian. (This implies that the algebra is not symmetric. See Theorem $A(a)$.) Later Gelfand and Neumark [3] extended this example to include all complex unimodular groups. Their proofs are quite difficult, entailing a knowledge of the irreducible unitary representations of the groups and considerable computation. Except for finite and commutative groups, the corresponding problems have not been studied for discrete groups. These problems will be our concern.

The main results will be summarized now. In § 2 several facts (some of which are well known) are collected to be used later. § 3 is concerned with the construction of group algebras that are symmetric, or at least have an hermitian involution. It is shown (Corollary 3.4) that the group algebra of the direct product of a commutative group and a group whose group algebra is symmetric, is a symmetric algebra. Theorem 3.7 shows that the natural involution is hermitian in the group algebra of a semidirect product of a commutative group by a finite group.

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