ON INVARIANT PROBABILITY MEASURES II

J. R. BLUM AND D. L. HANSON

1. **Summary.** We continue the work begun in [1]. In this paper we investigate convergence properties of sequences of probability measures which are asympototically invariant.

2. **Introduction.** Let Ω be a set, $\mathscr{A}$ be a σ-algebra of subsets of Ω, and T be a mapping of Ω onto Ω which is one-to-one and bimeasurable. A set $A \in \mathscr{A}$ is said to be invariant if $A = TA$, a probability measure Q defined on $\mathscr{A}$ is invariant if $Q(A) = Q(TA)$ for all $A \in \mathscr{A}$, and an invariant probability measure P is said to be ergodic if every invariant set A is trivial for P, i.e., if $P(A) = 0$ or $P(A) = 1$. Alternately an invariant probability measure P is ergodic if whenever $P(A) > 0$ we have

$$P\Big(\bigcup_{n=-\infty}^{\infty} T^n A\Big) = 1 .$$

Let $\{Q_n\}$ be a sequence of probability measures defined on $\mathscr{A}$. We shall say that the sequence is asymptotically invariant if $\lim_n [Q_n(A) - Q_n(TA)] = 0$ for every $A \in \mathscr{A}$. In § 3 we give a simple condition which yields convergence of such a sequence to a given ergodic measure. In § 4 an example is given which shows that a reasonable conjecture is in fact false, and further conditions are given which insure uniform convergence of a sequence of asymptotically invariant measures. In the last section we investigate convergence properties of certain sequences of probability density functions.

Throughout the paper we shall have occasion to refer to the following theorem, proved in [1]. We state it here as:

THEOREM 1. *If P and Q are invariant measures which agree on the invariant sets then $P = Q$.*

3. **A convergence theorem.** Let P be an ergodic measure (we shall assume throughout that every measure considered is a probability measure) and let Q be a measure absolutely continuous with respect to P. Define the sequence $\{Q_n\}$ for $n = 1, 2, \cdots$ by the formula

$$Q_n(A) = \frac{1}{n}\sum_{i=0}^{n-1} Q(T^i A) , \quad A \in \mathscr{A}$$

Then it is an immediate consequence of the individual ergodic theorem that $\lim_n Q_n(A) = P(A)$ for every $A \in \mathscr{A}$. Clearly the sequence $\{Q_n\}$ is

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