HARDY'S INEQUALITY AND ITS EXTENSIONS

PAUL R. BEESACK

1. Introduction. In this paper we are concerned with a systematic and uniform treatment of some analogues and extensions of Hardy's inequality for integrals. This result we state as

THEOREM 1. *If* $p > 1$, $f(x) \geqq 0$, *and* $F(x) = \int_0^x f(t)dt$, *then*

$$\int_0^\infty \left(\frac{F}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p dx$$

unless $f \equiv 0$. *The constant is the best possible.*

This theorem was first proved by Hardy [1], and various alternative proofs have been given by other authors. (For reference to these, see [3, 240—243].) Theorem 1, together with the following generalization of this result (also due to Hardy, [2] and [3, Th. 330]) may be regarded as models of the class of inequalities with which this paper deals.

THEOREM 2. *If* $p > 1$, $r \neq 1$, $f(x) \geqq 0$, *and* $F(x)$ *is defined by*

$$F(x) = \begin{cases} \int_0^x f(t)dt & (r > 1)\,, \\ \int_x^\infty f(t)dt & (r < 1)\,, \end{cases}$$

then

$$\int_0^\infty x^{-r} F^p dx < \left(\frac{p}{|r-1|}\right)^p \int_0^\infty x^{-r}(xf)^p dx$$

unless $f \equiv 0$. *Again the constant is the best possible.*

Our integral inequalities will be of the form

$$\int_a^b s(x) F^p dx \leqq \int_a^b r(x) f^p dx \tag{1.1}$$

where $p > 1$ (or $p < 0$), and F is defined (as in Theorem 2) as a suitable integral of $f(x)$. For $0 < p < 1$, we obtain inequalities of the form (1.1), but with the inequality sign reversed. Our method of proof differs from those referred to above. We make use of the Euler-Lagrange differential equations

Received July 14, 1959.