# OPERATIONAL CALCULUS OF LINEAR RELATIONS

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**1. Introduction.** Let $X$ and $Y$ be linear spaces, and $T$ a linear subspace of $X \oplus Y$. We call $T$ a *linear relation* to indicate our interest in those constructions with $T$ which generalize those carried out when $T$ is single-valued [4].

Properly many-valued linear relations arise naturally from operators $T$ when $T^{-1}$ or $T^*$ is contemplated in cases where they are not single-valued. One advantage of not dismissing $T^*$ when it is not single-valued is that $T^{**} = T$ if and only if $T$ is closed (for the details, see 3.34, below.) A more superficial attraction is that linear relations, even self-adjoint linear relations in Hilbert space can exhibit phenomena (unbounded spectrum, domain $\neq X$) in finite-dimensional spaces which linear operators exhibit only in infinite-dimensional spaces.

We present an outline of the paper. In § 2 we define $p(T)$ where $p$ is a polynomial with coefficients in the field $\Phi$ involved in $X$. We prove that $(pq)(T) = p(T)q(T)$, $(p \circ q)(T) = p(q(T))$, and point out that sometimes $(p + q)(T) \neq p(T) + q(T)$, etc.

In § 3 we turn to relations in dual pairs. In this situation, adjoints can be defined. We build an automorphism $\lambda \to \bar{\lambda}$ of $\Phi$ into the theory of dual pairs, so as not to *exclude* the Hilbert space situation, which dual pairs are intended to imitate. (Thus the transpose is a special kind of adjoint.) Closedness is defined algebraically, but in a way compatible with the topological concept. Closure of $T^*$ and other algebraic properties of * are established. Finally, it is shown that if $T$ is closed and its resolvent is not void then $p(T)$ is also closed.

Section 4 considers the self-dual case. We give a simple condition (4.3) always true in Hilbert space, that $T^*T$ be self-adjoint, $T$ being closed. In § 5 we give the spectral analysis of self-adjoint linear relations in Hilbert space. In a 1 : 1 manner these correspond to the unitary *operators*, via the Cayley transform. However, it can be shown directly that $X$ is the direct sum of orthogonal subspaces $Y, Z$ which reduce $T$ $(= T^*)$ giving in $Z$ a self-adjoint operator and in $Y$ the inverse of the zero-operator.

**2. Linear relations.** A *relation* $T$ between members of a set $X$ and members of a set $Y$ is merely a subset of $X \times Y$. For $x \in X$, $T(x) = \{y : (x, y) \in T\}$. The *domain* of $T$ consists of those $x$ such that $T(x)$ is not void. $T$ is called single-valued if $T(x)$ never contains more than one element. The *range* of $T$ is the union of all $T(x)$.

Received April 13, 1960.