THE STRUCTURE OF CERTAIN MEASURE ALGEBRAS

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Introduction. In their paper [3], Hewitt and Zuckerman study the measure algebra $\mathscr{M}(G)$ where G is a topological semigroup of the following type: G is a linearly ordered set topologized with the order topology, is compact in this topology, and multiplication is defined by $xy = \max(x, y)$. In this study, we will suppose that G has the above properties except that compactness will be replaced by local compactness. (See § 8.5 [3]). As the reader will readily observe, we are heavily indebted to Hewitt and Zuckerman for their initial study of these measure algebras. For completeness, we have listed, without proof, a few of their results; they are stated in their paper for compact semigroups but the proofs easily carry over to locally compact semigroups.

In § 2 we study $\hat{G}$ and $\hat{G}_0$. The characterization of the Gel'fand topology on $\hat{G}$ is somewhat simpler than that of Theorem 5.5 [3]. The major result of this study is Theorem 3.4, stating that every closed ideal in $\mathscr{M}(G)$ is the intersection of maximal ideals; i.e., spectral synthesis holds for $\mathscr{M}(G)$. Malliavin [7] has recently shown that spectral synthesis fails for $\mathscr{M}(G)$ when G is a non-compact locally compact commutative group.[1] Theorem 3.4 shows that this result cannot be generalized to locally compact commutative semigroups. In § 4, a generalization of Theorem 6.7 [3] is indicated; see Theorem 4.5. This is used to obtain additional facts about $\mathscr{M}(G)$ (§ 5). In 5.8 we show that our theory is not a special case of the theory of function algebras.

1. Preliminaries.

1.1. We will be concerned with linearly ordered sets; i.e. sets ordered by transitive, irreflexive relations $<$. For elements x and y in such a set X, we define $]x, y[= \{z \in X : x < z < y\}$ and $[x, y] = \{z \in X : x \leq z \leq y\}$. The half-open intervals $[x, y[$ and $]x, y]$ are defined analogously. We also define $]-\infty, x[= \{z \in X : z < x\}$ and $]-\infty, x] = \{z \in X : z \leq x\}$ with analogous definitions for $[x, \infty[$, $]x, \infty[$, and $]-\infty, \infty[$. The symbols $-\infty$ and ∞ will never denote actual elements of X. The order topology for X is the topology having the family $\{]-\infty, x[\}_{x \in X} \cup$

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[1] Actually Malliavin shows that spectral synthesis fails for $L_1(G)$; the result for $\mathscr{M}(G)$ follows easily from this.